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On (N, ϵ)-pseudospectra of operators on Banach spaces

Markus Seidel *

Chemnitz University of Technology, Reichenhainer Str. 39, 08126 Chemnitz, Germany

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Abstract

In this paper we extend the concept of the (N, ϵ)-pseudospectra of Hansen to the case of bounded linear operators on Banach spaces and prove several relations to the usual spectrum. We particularly discuss the approximation by rectangular finite sections and the impact of the fundamental result of Shargorodsky on "jumping" pseudospectra.

Keywords: Spectrum; Pseudospectrum; Banach space; Quasi-diagonal operator

The (N, ϵ)-pseudospectrum

In [11] the authors write "A computer working with finite accuracy cannot distinguish between a non-invertible matrix and an invertible matrix the inverse of which has a very large norm". Therefore one replaces the spectrum $\operatorname{sp} A$ of a bounded linear operator $A \in \mathcal{L}(\mathbf{X})$ on a Banach space $\mathbf{X}$ by the so-called pseudospectra which reflect finite accuracy. For some pioneering work on that topic we refer to Landau [16,17], Reichel and Trefethen [21] and Böttcher [3], and particularly point out the comprehensive monograph [25] of Trefethen and Embree.

Definition 1. For $N \in \mathbb{N}_0$ and $\epsilon > 0$ the (N, ϵ)-pseudospectrum of a bounded linear operator A on a complex Banach space $\mathbf{X}$ is defined as the set

$$\operatorname{sp}_{N,\epsilon} A := \left\{z \in \mathbb{C}\colon \left\|(A - zI)^{-2^N}\right\|^{2^{-N}} \geqslant 1/\epsilon\right\}.^1$$

* Fax: +49 37153122509.
E-mail address: markus.seidel@mathematik.tu-chemnitz.de.
[1] In this text we use the convention $\|B^{-1}\| = \infty$ if B is not invertible.

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Notice that for $N = 0$ this is the definition of the (classical) ϵ-pseudospectra which have gained attention after it was discovered in [21] and [3] that they approximate the spectrum but are less sensitive to perturbations, and moreover the ϵ-pseudospectra of discrete convolution operators mimic exactly the ϵ-pseudospectrum of an appropriate limiting operator, which is in general not true for the "usual" spectrum. See also [2,5,11,4] and the references cited there.

Later on, Hansen [13,14] introduced the more general (N, ϵ)-pseudospectra for linear operators on separable Hilbert spaces and pointed out that they share several nice properties with case $N = 0$, but offer a better insight into the approximation of the spectrum. More precisely, his result tells that the spectrum of an operator can be approximated by its (N, ϵ)-pseudospectra with respect to the Hausdorff distance[2] d_H and can be stated as follows, where the closed ϵ-neighborhood of a set $S \subset \mathbb{C}$ is denoted by $B_\epsilon(S) := \{z \in \mathbb{C}\colon \operatorname{dist}(z, S) \leqslant \epsilon\}$.

Theorem 2. *Let $A \in \mathcal{L}(\mathbf{X})$. For every $\delta > \epsilon > 0$ there is an N_0 such that, for all $N \geqslant N_0$,*

$$B_\epsilon(\operatorname{sp} A) \subset \operatorname{sp}_{N,\epsilon} A \subset B_\delta(\operatorname{sp} A). \tag{1}$$

Furthermore, Hansen discussed how the pseudospectrum can be approximated numerically, based on the consideration of singular values of certain finite matrices. In a very recent preprint [15] he and Nevanlinna pointed out that the Banach space version of Theorem 2 is in force, but the mentioned Hilbert space approach for the approximate determination of the (N, ϵ)-pseudospectrum via singular values cannot be extended to the Banach space case since there is no involution available anymore. Therefore we propose a modification which replaces the singular values by the injection and surjection modulus. Here the precise description comes:

Rectangular finite sections and their contribution to the approximation of spectra

For bounded linear operators $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ on Banach spaces $\mathbf{X}, \mathbf{Y}$ we denote by $B_{\mathbf{X}}$ the closed unit ball in $\mathbf{X}$ and by

$$j(A) := \sup\bigl\{\tau \geqslant 0\colon\ \|Ax\| \geqslant \tau\|x\| \text{ for all } x \in \mathbf{X}\bigr\},$$
$$q(A) := \sup\bigl\{\tau \geqslant 0\colon\ A(B_{\mathbf{X}}) \supset \tau B_{\mathbf{Y}}\bigr\}$$

the injection modulus and the surjection modulus, respectively. Due to [19, B.3.8], it holds that

$$j(A) = \inf\bigl\{\|Ax\|\colon\ x \in \mathbf{X},\ \|x\| = 1\bigr\}, \quad j\bigl(A^*\bigr) = q(A) \text{ and } q\bigl(A^*\bigr) = j(A). \tag{2}$$

Hence $j, q : \mathcal{L}(\mathbf{X}) \to \mathbb{R}$ are continuous. Furthermore, we have $j(A) = q(A) = \|A^{-1}\|^{-1}$ if A is invertible, and $j(A)$ ($q(A)$) equals zero if A is not invertible from the left (right, respectively). From these facts we now easily derive another observation which proves to be extremely helpful in the subsequent considerations.

Lemma 3. *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be Banach spaces and $A : \mathbf{X} \to \mathbf{Y}$ as well as $B : \mathbf{Y} \to \mathbf{Z}$ be bounded linear operators. Then $j(BA) \geqslant j(B)\,j(A)$ and $q(BA) \geqslant q(B)\,q(A)$.*

[2] For $S, T \subset \mathbb{C}$ compact, $d_H(S, T) = \max\{\max_{s\in S} \operatorname{dist}(s, T), \max_{t\in T} \operatorname{dist}(t, S)\}$, where $\operatorname{dist}(s, T) = \min_{t\in T} |s - t|$.