Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 256 (2009) 3236–3256
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Rough path limits of the Wong–Zakai type with a modified drift term

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Received 5 August 2008; accepted 17 February 2009

Available online 9 March 2009

Communicated by Paul Malliavin

Abstract

The Wong–Zakai theorem asserts that ODEs driven by "reasonable" (e.g. piecewise linear) approximations of Brownian motion converge to the corresponding Stratonovich stochastic differential equation. With the aid of rough path analysis, we study "non-reasonable" approximations and go beyond a well-known criterion of [Ikeda, Watanabe, North Holland, 1989] in the sense that our result applies to perturbations on all levels, exhibiting additional drift terms involving any iterated Lie brackets of the driving vector fields. In particular, this applies to the approximations by McShane ('72) and Sussmann ('91). Our approach is not restricted to Brownian driving signals. At last, these ideas can be used to prove optimality of certain rough path estimates.

Keywords: Iterated Lie brackets in limit processes of differential equations; Rough paths analysis

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[1] Partially supported by a Leverhulme Research Fellowship and EPSRC Grant EP/E048609/1.
[2] Partially supported by EPSCR Grant EP/P502365/1 and a DOC-fellowship of the Austrian Academy of Sciences.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.02.010

1. Preliminaries

1.1. Rough differential equations

Let $\alpha \in (0,1]$. A weak geometric α-Hölder rough path $\mathbf{x}$ over $\mathbb{R}^d$ is a continuous path on $[0,T]$ with values in $G^{[1/\alpha]}(\mathbb{R}^d)$, the step[3]-$[1/\alpha]$ nilpotent group over $\mathbb{R}^d$, of finite α-Hölder regularity relative to d, the Carnot–Carathéodory metric on $G^{[p]}(\mathbb{R}^d)$, i.e.

$$\|\mathbf{x}\|_{\alpha\text{-Höl};[0,T]} = \sup_{0 \leqslant s < t \leqslant T} \frac{d(\mathbf{x}_s, \mathbf{x}_t)}{|t-s|^\alpha} < \infty.$$

For orientation, let us discuss the case $\alpha \in (1/3, 1/2)$, which covers Brownian motion (for details see [5,11,17,18]). We realize $G^2(\mathbb{R}^d)$ as the set of all $(a,b) \in \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$ for which $Sym(b) \equiv a^{\otimes 2}/2$. (This point of view is natural: a smooth $\mathbb{R}^d$-valued path $x = (x_t^i)_{i=1,\dots,d}$, enhanced with its iterated integrals $\int_0^t \int_0^s \mathrm{d}x_u^i \,\mathrm{d}x_s^j$, gives canonically rise to a $G^2(\mathbb{R}^d)$-valued path.). Given $(a,b) \in G^2(\mathbb{R}^d)$ one gets rid of the redundant $Sym(b)$ by $(a,b) \mapsto (a, b - a^{\otimes}/2) \in \mathbb{R}^d \oplus so(d)$. Applied to x enhanced with its iterated integrals over $[0,t]$ this amounts to look at the path x and its (signed) areas $\int_0^t x_{0,s}^i \,\mathrm{d}x_s^j - \int_0^t x_{0,s}^j \,\mathrm{d}x_s^i$, $i,j \in \{1,\dots,d\}$.[4] Without going in too much detail, the group structure on $G^2(\mathbb{R}^d)$ can be identified with the (truncated) tensor multiplication and is relevant as it allows to relate algebraically the path and area increments over adjacent intervals; the mapping $(a,b) \mapsto (a, b - a^{\otimes}/2)$ maps the Lie group $G^2(\mathbb{R}^d)$ to its Lie algebra $\mathfrak{g}^2(\mathbb{R}^d)$; at last, the Carnot–Carathéodory metric is defined intrinsically as (left-) invariant metric on $G^2(\mathbb{R}^d)$ and satisfies $|a| + |b|^{1/2} \lesssim d((0,0),(a,b)) \lesssim |a| + |b|^{1/2}$.

One can then think of a geometric α-Hölder rough path $\mathbf{x}$ as a path $x:[0,T] \to \mathbb{R}^d$ enhanced with its iterated integrals (equivalently: area integrals) although the later need not make classical sense. For instance, *almost every* joint realization of Brownian motion and Lévy's area process is a geometric α-Hölder rough path. Lyons' theory of rough paths then gives deterministic meaning to the rough differential equation (RDE)

$$\mathrm{d}y = V(y)\,\mathrm{d}\mathbf{x}, \qquad y(0) = y_0$$

for Lip^Γ-vector fields (in the sense of Stein[5]), $\Gamma > 1/\alpha \geqslant 1$, and we write $y_t = \pi(0, y_0; x)_t$ for this solution. By considering the space–time rough path $\tilde{\mathbf{x}} = (t, \mathbf{x})$ and $\tilde{V} = (V_0, V_1, \dots, V_d)$ one can consider RDEs with drift. Although well studied [16], with a view towards minimal regularity assumptions on V_0, we shall need certain "Euler" estimates [8] for RDEs with drift which are not available in the current literature. The "Doss–Sussmann method" (implemented for RDEs in Section 3) will provide a quick route to these estimates.

1.2. Perturbed rough paths and iterated Lie brackets

Assume we are given a weak geometric α-Hölder rough path $\mathbf{x}$ and a path $\mathbf{p}$ that takes values in the center of $G^N(\mathbb{R}^d)$, N some integer $N \geqslant [1/\alpha]$ (think of the path $\mathbf{p}$ as a perturbation of our

[3] $[\cdot]$ gives the integer part of a real number.

[4] Given an interval $I = [a,b]$, for brevity we write $x_I \equiv x_{a,b} \equiv x_b - x_a$.

[5] I.e. a function is Lip^γ if it is $\lfloor\gamma\rfloor$-times ($\lfloor . \rfloor = [.] - 1$ on integers, otherwise equal) differentiable, the $\lfloor\gamma\rfloor$th derivative is $\gamma - \lfloor\gamma\rfloor$-Hölder continuous and the function and all its derivatives are bounded.