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Explicit and almost explicit spectral calculations for diffusion operators

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Abstract

The diffusion operator

$$H_{\mathrm{D}} = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} - b\frac{d}{dx} = -\frac{1}{2}\exp(-2B)\frac{d}{dx}a\exp(2B)\frac{d}{dx},$$

where $B(x) = \int_0^x \frac{b}{a}(y)\,dy$, defined either on $\mathbb{R}^+ = (0,\infty)$ with the Dirichlet boundary condition at $x = 0$, or on $\mathbb{R}$, can be realized as a self-adjoint operator with respect to the density $\exp(2Q(x))\,dx$. The operator is unitarily equivalent to the Schrödinger-type operator $H_{\mathrm{S}} = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + V_{b,a}$, where $V_{b,a} = \frac{1}{2}(\frac{b^2}{a} + b')$. We obtain an explicit criterion for the existence of a compact resolvent and explicit formulas up to the multiplicative constant 4 for the infimum of the spectrum and for the infimum of the essential spectrum for these operators. We give some applications which show in particular how $\inf\sigma(H_{\mathrm{D}})$ scales when $a = \nu a_0$ and $b = \gamma b_0$, where ν and γ are parameters, and a_0 and b_0 are chosen from certain classes of functions. We also give applications to self-adjoint, multi-dimensional diffusion operators.

Keywords: Spectrum; Essential spectrum; Compact resolvent; Diffusion; Schrödinger operator

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1. Introduction and statement of results

In this paper, we give an explicit formula up to the multiplicative constant 4 for the bottom of the spectrum and for the bottom of the essential spectrum for diffusion operators on the half-line $\mathbb{R}^+ = (0, \infty)$ with the Dirichlet boundary condition at 0, and for diffusion operators on the entire line. Assuming a little more regularity, each such operator is unitarily equivalent to a certain Schrödinger-type operator, so we also obtain the same information for these latter operators. Recall that such an operator possesses a compact resolvent if and only if its essential spectrum is empty, or equivalently, if and only if the infimum of its essential spectrum is ∞. Thus, we obtain a completely explicit criterion for the existence of a compact resolvent. A diffusion operator with a compact resolvent is particularly nice because its transition (sub-)probability density $p(t, x, y)$ (with respect to the reversible measure) can be written in the form $p(t, x, y) = \sum_{n=0}^{\infty} \exp(-\lambda_n t)\phi_n(x)\phi_n(y)$, where $\{\phi_n\}_{n=0}^{\infty}$ is a complete, orthonormal set of eigenfunctions and $\{\lambda_n\}_{n=0}^{\infty}$, satisfying $0 \leqslant \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, are the corresponding eigenvalues. We give some applications of the results, which show in particular how $\inf \sigma(H_\mathrm{D})$ scales when $a = \nu a_0$ and $b = \gamma b_0$, where ν and γ are parameters, and a_0 and b_0 are chosen from certain classes of functions. At the end of the paper, we give applications to self-adjoint, multi-dimensional diffusion operators of the form $-\frac{1}{2}\nabla \cdot a\nabla - a\nabla Q \cdot \nabla = -\frac{1}{2}\exp(-2Q)\nabla \cdot a\exp(2Q)\nabla$ on $L^2(\mathbb{R}^d, \exp(2Q)\,dx)$. The methods and the statements of the results are analytic, but many of the formulas and results have probabilistic import.

We begin with the theory on the half-line, wherein lies the crux of our method. The results for the entire line follow readily from the results for the half-line. Let $0 < a \in C^1([0, \infty))$ and $b \in C([0, \infty))$. Define $B(x) = \int_0^x \frac{b}{a}(y)\,dy$. Consider the diffusion operator with divergence-form diffusion coefficient a and drift b

$$H_\mathrm{D} = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} - b\frac{d}{dx} = -\frac{1}{2}\exp(-2B)\frac{d}{dx}a\exp(2B)\frac{d}{dx}$$

on $\mathbb{R}^+$ with the Dirichlet boundary condition at $x = 0$. One can realize H_D as a non-negative, self-adjoint operator on $L^2(\mathbb{R}^+, \exp(2B)\,dx)$ via the Friedrichs extension of the closure of the non-negative quadratic form

$$Q_\mathrm{D}(f, g) = \frac{1}{2}\int_0^\infty (f'ag')\exp(2B)\,dx,$$

defined for $f, g \in C_0^1(\mathbb{R}^+)$, the space of continuously differentiable functions with compact support on $\mathbb{R}^+$.

Let U_B denote the unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+, \exp(2B)\,dx)$ defined by

$$U_B f = \exp(-B) f.$$

Assuming that $b \in C^1(\mathbb{R}^+)$, define $H_\mathrm{S} = U_B^{-1} H_\mathrm{D} U_B$. One can check that

$$H_\mathrm{S} = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + V_{b,a},$$

where