Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Functional Analysis 256 (2009) 3368–3408
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Compact quantum metric spaces and ergodic actions of compact quantum groups

Hanfeng Li [1]

Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260, USA

Received 2 September 2008; accepted 9 September 2008

Available online 25 September 2008

Communicated by Alain Connes

Abstract

We show that for any co-amenable compact quantum group $A = C(\mathcal{G})$ there exists a unique compact Hausdorff topology on the set $\mathrm{EA}(\mathcal{G})$ of isomorphism classes of ergodic actions of $\mathcal{G}$ such that the following holds: for any continuous field of ergodic actions of $\mathcal{G}$ over a locally compact Hausdorff space T the map $T \to \mathrm{EA}(\mathcal{G})$ sending each t in T to the isomorphism class of the fibre at t is continuous if and only if the function counting the multiplicity of γ in each fibre is continuous over T for every equivalence class γ of irreducible unitary representations of $\mathcal{G}$. Generalizations for arbitrary compact quantum groups are also obtained. In the case $\mathcal{G}$ is a compact group, the restriction of this topology on the subset of isomorphism classes of ergodic actions of full multiplicity coincides with the topology coming from the work of Landstad and Wassermann. Podleś spheres are shown to be continuous in the natural parameter as ergodic actions of the quantum $\mathrm{SU}(2)$ group. We also introduce a notion of regularity for quantum metrics on $\mathcal{G}$, and show how to construct a quantum metric from any ergodic action of $\mathcal{G}$, starting from a regular quantum metric on $\mathcal{G}$. Furthermore, we introduce a quantum Gromov–Hausdorff distance between ergodic actions of $\mathcal{G}$ when $\mathcal{G}$ is separable and show that it induces the above topology.

Keywords: Compact quantum group; Ergodic action; Continuous field; Compact quantum metric space

E-mail address: hfli@math.buffalo.edu.

[1] Partially supported by NSF Grant DMS-0701414.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.09.009

1. Introduction

An ergodic action of a compact group G on a unital C^*-algebra B is a strongly continuous action of G on B such that the fixed point algebra consists only of scalars. For an irreducible representation of G on a Hilbert space H, the conjugate action of G on the algebra $B(H)$ is ergodic. On the other hand, ergodic actions of G on commutative unital C^*-algebras correspond exactly to translations on homogeneous spaces of G. Thus the theory of ergodic actions of G connects both the representation theory and the study of homogeneous spaces. See [14,20,26, 42–44] and references therein.

Olesen, Pedersen, and Takesaki classified faithful ergodic actions of an abelian compact group as skew-symmetric bicharacters on the dual group [26]. Landstad and Wassermann generalized their result independently to show that ergodic actions of full multiplicity of an arbitrary compact group G are classified by equivalence classes of dual cocycles [20,43]. However, the general case is quite difficult—so far there is no classification of (faithful) ergodic actions of compact groups, not to mention compact quantum groups. In this paper we are concerned with topological properties of the whole set $\mathrm{EA}(G)$ of isomorphism classes of ergodic actions of a compact group G, and more generally, the set $\mathrm{EA}(\mathcal{G})$ of isomorphism classes of ergodic actions of a compact quantum group $A = C(\mathcal{G})$.

As a consequence of their classification, Olesen, Pedersen, and Takesaki showed that the set of isomorphism classes of faithful ergodic actions of an abelian compact group has a natural abelian compact group structure. From the work of Landstad and Wassermann, the set $\mathrm{EA}(G)_{\mathrm{fm}}$ of ergodic actions of full multiplicity of an arbitrary compact group G also carries a natural compact Hausdorff topology.

There are many ergodic actions not of full multiplicity, such as conjugation actions associated to irreducible representations and actions corresponding to translations on homogeneous spaces (unless G is finite or the homogeneous space is G itself). In the physics literature concerning string theory and quantum field theory, people talk about fuzzy spheres, the matrix algebras $M_n(\mathbb{C})$, converging to the two-sphere S^2 (see the introduction of [36] and references therein). One important feature of this convergence is that each term carries an ergodic action of $\mathrm{SU}(2)$, which is used in the construction of this approximation of S^2 by fuzzy spheres. Thus if one wants to give a concrete mathematical foundation for this convergence, it is desirable to include the $\mathrm{SU}(2)$ symmetry. However, none of these actions involved are of full multiplicity, and hence the topology of Landstad and Wassermann does not apply here.

For compact quantum groups there are even more interesting examples of ergodic actions, see [41]. Podleś introduced a family of quantum spheres S^2_{qt}, parameterized by a compact subset T_q of the real line, as ergodic actions of the quantum $\mathrm{SU}(2)$ group $\mathrm{SU}_q(2)$ satisfying certain spectral conditions [29]. These quantum spheres carry interesting non-commutative differential geometry [7,8]. One also expects that Podleś quantum spheres are continuous in the natural parameter t as ergodic actions of $\mathrm{SU}_q(2)$.

Continuous change of C^*-algebras is usually described qualitatively as continuous fields of C^*-algebras over locally compact Hausdorff spaces [9, Chapter 10]. There is no difficulty to formulate the equivariant version—continuous fields of actions of compact groups [32] or even compact quantum groups (see Section 5 below). Thus if there is any natural topology on $\mathrm{EA}(\mathcal{G})$, the relation with continuous fields of ergodic actions should be clarified.

One distinct feature of the theory of compact quantum groups is that there is a full compact quantum group and a reduced compact quantum group associated to each compact quantum group $\mathcal{G}$, which may not be the same. A compact quantum group $\mathcal{G}$ is called co-amenable if