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A generalization of sectorial and quasi-sectorial operators

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Abstract

In the paper we generalize the main results presented in Bentkus and Paulauskas (2004) [2] by giving rates of approximation of some semigroups of operators of the order $n^{-\alpha}$, $0 < \alpha \leqslant 1$. Also two classes of operators, generalizing sectorial and quasi-sectorial operators, are introduced and their properties are studied.

Keywords: Chernoff theory; Sectorial operators; Ritt condition

1. Introduction and formulation of results

It is well known that the famous Chernoff "$\sqrt{n}$-lemma" and its extensions play a rather important role in the semigroup theory of operators, particulary in approximation problems for semigroups of operators. We recall this result. Let A be a linear contraction of a Banach space X. Then $e^{t(A-I)}$, $t \geqslant 0$, is a contraction semigroup, and

$$\left\| A^n x - e^{n(A-I)} x \right\| \leqslant n^{1/2} \| Ax - Ix \|, \quad \text{for all } x \in X,$$

(see [5], Lemma 2).

Recently in the paper [2] (see Theorem 1.1 there) the following extension of this result was proved. Let A be an operator on a Banach space X and let $B = I - A$. We consider the condition

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$$(n+1)\left\|tB(I-tB)^n\right\| \leqslant K, \quad \text{for all } 0 \leqslant t \leqslant 1,\ n=0,1,2,\ldots \tag{1}$$

with some constant K independent of t and n.

Theorem A. *Assume that A is a contraction of a Banach space X and satisfies condition* (1). *Then we have*

$$\Delta_{n,s} := \left\|(I-sB/n)^n - e^{-sB}\right\| \leqslant 4K^2/n, \tag{2}$$

for all $0 \leqslant s \leqslant n$ and $n = 1, 2, \ldots$. In particular, with $s = n$, we have

$$\Delta_n := \left\|A^n - e^{n(A-I)}\right\| \leqslant 4K^2/n, \tag{3}$$

for all $n = 1, 2, \ldots$.

This result is optimal, since it is not difficult to show that for a real x and $f(x) = |x^n - \exp(n(x-1))|$ we have $\max_{0\leqslant x\leqslant 1} f(x) > c/n$, with some positive constant $c > 0$.

In the case $t = 1$ condition (1) becomes

$$(n+1)\left\|A^n - A^{n+1}\right\| \leqslant K. \tag{4}$$

For a Banach space operator A consider the condition

$$\left\|(A-\lambda I)^{-1}\right\| \leqslant c|\lambda-1|^{-1}, \quad \text{for } |\lambda| > 1,\ \lambda \in \mathbb{C}, \tag{5}$$

where c is a constant and $\mathbb{C}$, as usual, denote the complex plane. Condition (5) is called the Ritt condition. Some authors (see, for example, [21]) call this condition as the Tadmor–Ritt condition, for the reason that this condition in the above written form was introduced in [20], while the Ritt original condition [19] seemed a little bit weaker. But in [3] it was shown that these two conditions are equivalent. It is known (see [13] and [16]), that the Ritt condition is equivalent to (4) combined with the power boundedness condition $\sup_n \|A^n\| < \infty$. Also it is worth to mention that the Ritt condition is connected with some other problems of the operator theory, for example, with the Gelfand–Hille theorem, see the survey paper [23].

It was noted in [2] that the conditions of Theorem A are equivalent to the Ritt condition. The result of Theorem A generalizes and strengthens some previously known results from [4,17].

Also in [2] there was proved the following theorem giving an optimal error bound for the Euler approximations of semigroups of operators in Banach spaces.

Theorem B. *Let e^{-tA}, $t \geqslant 0$, be a semigroup of operators in a Banach space. Assume that there exists a constant K independent of n and t such that*

$$n\left\|tA(I+tA)^{-n}\right\| \leqslant K, \tag{6}$$

and

$$\left\|e^{-tA}\right\| \leqslant K, \qquad \left\|tAe^{-tA}\right\| \leqslant K, \tag{7}$$