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Journal of Functional Analysis 262 (2012) 2481–2515

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

The Feller property on Riemannian manifolds

Stefano Pigola, Alberto G. Setti *

Sezione di Matematica – DiSAT, Università dell'Insubria – Como, via Valleggio 11, I-22100 Como, Italy

Received 24 May 2011; accepted 4 December 2011

Available online 21 December 2011

Communicated by B. Driver

Abstract

The asymptotic behavior of the heat kernel of a Riemannian manifold gives rise to the classical concepts of parabolicity, stochastic completeness (or conservative property) and Feller property (or C^0-diffusion property). Both parabolicity and stochastic completeness have been the subject of a systematic study which led to discovering not only sharp geometric conditions for their validity but also an incredible rich family of tools, techniques and equivalent concepts ranging from maximum principles at infinity, function theoretic tests (Khas'minskii criterion), comparison techniques etc. The present paper aims to move a number of steps forward in the development of a similar apparatus for the Feller property.

Keywords: Feller property; Riemannian manifolds; Comparison results

0. Introduction

This paper is a contribution to the theory of Riemannian manifolds satisfying the Feller (or C_0-diffusion) property for the Laplace–Beltrami operator. Since the appearance of the beautiful, fundamental paper by R. Azencott [2], new insights into such a theory (for the Laplace operator) are mainly confined into some works by S.T. Yau [27], J. Dodziuk [10], P. Li and L. Karp [19], E. Hsu [16,17], E.B. Davies [9]. These papers, which have been extended to more general classes of diffusion operators (see e.g. [25,18]) are devoted to the search of optimal geometric conditions for a manifold to enjoy the Feller property. In fact, with the only exception of Davies', the geo-

* Corresponding author.
E-mail addresses: stefano.pigola@uninsubria.it (S. Pigola), alberto.setti@uninsubria.it (A.G. Setti).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.12.001

metric conditions are always subsumed to Ricci curvature lower bounds. The methods employed to reach their results range from estimates of solutions of parabolic equations (Dodziuk, Yau, Li, Karp) up to estimates of the probability of the Brownian motion on M to be found in certain regions before a fixed time (Hsu). The probabilistic approach, which led to the best known condition on the Ricci tensor, relies on a result by Azencott (see also [17]) according to which *M is Feller if and only if, for every compact set K and for every $t_0 > 0$, the Brownian motion X_t of M issuing from x_0 enters K before the time t_0 with a probability that tends to zero as $x_0 \to \infty$.*

Our point of view will be completely deterministic and, although parabolic equations will play a key role in a number of crucial sections, it will most often depend on elliptic equation theory.

Beside, and closely related, to the Feller property one has the notions of parabolicity and stochastic completeness. Recall that M is said to be parabolic if every bounded above subharmonic function must be constant. Equivalently, the (negative definite) Laplace–Beltrami operator Δ of M does not possesses a minimal, positive Green kernel. From the probabilistic viewpoint, M is parabolic if the Brownian motion X_t enters infinitely many times a fixed compact set, with positive probability (recurrence). We also recall that M is conservative or stochastically complete if, for some (hence any) constant $\lambda > 0$, every bounded, positive λ-subharmonic function on M must be identically equal to 0. Here, λ-subharmonic means that $\Delta u \geqslant \lambda u$. Equivalently, M has the conservative property if the heat kernel of M has mass identically equal to 1. From the probabilistic viewpoint stochastic completeness means that, with probability 1, the Brownian paths do not explode to ∞ in a finite time. Clearly a parabolic manifold is stochastically complete.

Both parabolicity and stochastic completeness have been the subject of a systematic study which led to discovering not only sharp geometric conditions for their validity (in fact, volume growth conditions) but also an incredible rich family of tools, techniques and equivalent concepts ranging from maximum principles at infinity, function theoretic tests (Khas'minskii criterion), comparison techniques etc. The interested reader can consult e.g. the excellent survey paper by A. Grigor'yan [14]. See also [22,23] for the maximum principle perspective.

The present paper aims to move a number of steps forward in the development of a similar apparatus for the Feller property. Originally we also thought we would adopt an elliptic point of view. While, in many instances, this proves to be the most effective approach (for instance, in obtaining comparison results, or in the treatment of ends), in some cases it is not clear how to implement it, and we have to resort to the parabolic point of view (for instance studying minimal surfaces, or Riemannian coverings).

To make the treatment more readable, we decide to include the proofs of some of the basic results that are crucial in the development of the theory. Sometimes, we shall use a somewhat different perspective and more straightforward arguments. In fact, our attempt is to use a geometric slant from the beginning of the theory, most notably in interpreting the 1-dimensional case in terms of model manifolds.

Before giving a detailed survey of the organization of the paper, we would like to announce some future developments that, in our opinion, make the study of Feller manifolds even more stimulating. Indeed, quite naturally, using the Feller property enables one to study qualitative properties of solution of PDE's which are defined only in a neighborhood of infinity. This, in turn, has applications in a number of different geometric and analytic settings, e.g., in the theory of isometric immersions. This viewpoint will be extensively developed in a forthcoming paper [3].