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The reduced effect of a single scattering with a low-mass particle via a point interaction

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Abstract

In this article, we study a second-order expansion for the effect induced on a large quantum particle which undergoes a single scattering with a low-mass particle via a repulsive point interaction. We give an approximation with third-order error in λ to the map $G \to \mathrm{Tr}_2[(I \otimes \rho) S_\lambda^*(G \otimes I) S_\lambda]$, where $G \in B(L^2(\mathbb{R}^n))$ is a heavy-particle observable, $\rho \in B_1(\mathbb{R}^n)$ is the density matrix corresponding to the state of the light particle, $\lambda = \frac{m}{M}$ is the mass ratio of the light particle to the heavy particle, $S_\lambda \in B(L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n))$ is the scattering matrix between the two particles due to a repulsive point interaction, and the trace is over the light-particle Hilbert space. The third-order error is bounded in operator norm for dimensions one and three using a weighted operator norm on G.

Keywords: Scattering operator; Point interaction

1. Introduction

In theoretical physics, many derivations of decoherence models begin with an analysis of the effect on a test particle of a scattering with a single particle from a background gas [6,8,9]. A regime that the theorists have studied and which has generated interest in experimental physics [7] is when the test particle is much more massive than a single particle from the gas. Mathematical progress towards justifying the scattering assumption made in the physical literature in the regime where a test particle interacts with particles of comparatively low mass can

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be found in [1,3,5]. In this article, we study a scattering map expressing the effect induced on a test particle of mass M by an interaction with a particle of mass $m = \lambda M$, $\lambda \ll 1$. The force interaction between the test particle and the gas particle is taken as a repulsive point potential.

We work towards bounding the error $\epsilon(G, \lambda)$ in operator norm for $G \in B(L^2(\mathbb{R}^n))$, $n = 1, 3$ of a second order approximation:

$$\mathrm{Tr}_2\big[(I \otimes \rho)\mathbf{S}_\lambda^*(G \otimes I)\mathbf{S}_\lambda\big] = G + \lambda M_1(G) + \lambda^2 M_2(G) + \epsilon(G, \lambda), \tag{1.1}$$

where $\rho \in B_1(L^2(\mathbb{R}^n))$ is a density matrix (i.e. $\rho \geqslant 0$ and $\mathrm{Tr}[\rho] = 1$), $G \in B(L^2(\mathbb{R}^n))$, $\mathbf{S}_\lambda \in B(L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n))$ is the unitary scattering operator for a point interaction, and the partial trace is over the second component of the Hilbert space $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$. M_1 and M_2 are linear maps acting on a dense subspace of $B(L^2(\mathbb{R}^n))$ (M_2 is unbounded). Our main result is that there exists a $c > 0$ such that for all ρ, G, and $0 \leqslant \lambda$

$$\big\|\epsilon(G, \lambda)\big\| \leqslant c\lambda^3 \|\rho\|_{wtn} \|G\|_{wn},$$

where $\|\cdot\|_{wn}$ is a weighted operator norm of the form

$$\|G\|_{wn} = \|G\| + \big\||\vec{X}|G\big\| + \big\|G|\vec{X}|\big\| + \sum_{0 \leqslant i,j \leqslant d} \big(\|X_i P_j G\| + \|G P_j X_i\|\big) + \sum_{e_1+e_2 \leqslant 3} \big\||\vec{P}|^{e_1} G |\vec{P}|^{e_2}\big\|,$$

and $\|\cdot\|_{wtn}$ is a weighted trace norm which will depend on the dimension. In the above, $\vec{X}$ and $\vec{P}$ are the vector of position and momentum operators respectively: $(X_j f)(x) = x_j f(x)$ and $(P_j f)(x) = i(\frac{\partial}{\partial x_j} f)(x)$. Expressions of the type $A^* G B$ for unbounded operators A and B are identified with the kernel of the densely defined quadric form $F(\psi_1; \psi_2) = \langle A\psi_1 \mid G B \psi_2\rangle$ in the case that F is bounded.

The scattering operator is defined as $\mathbf{S}_\lambda = (\Omega^+)^* \Omega^-$, where

$$\Omega^\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH_{\mathrm{tot}}} e^{-itH_{\mathrm{kin}}} \tag{1.2}$$

are the Möller wave operators, and H_{kin} is the kinetic Hamiltonian and is the standard self-adjoint extension of the sum of the Laplacians $-\frac{1}{2M}\Delta_{\mathrm{heavy}} - \frac{1}{2m}\Delta_{\mathrm{light}}$, while the total Hamiltonian H_{tot} includes an additional repulsive point interaction between the particles. The definition of H_{tot} is a little tricky for $n > 1$ since, in analogy to the Hamiltonian for a particle in a point potential [2], it cannot be defined as a perturbation of H_{kin} even in the sense of a quadratic form. Rather, it is defined as a self-adjoint extension of $-\frac{1}{2M}\Delta_{\mathrm{heavy}} - \frac{1}{2m}\Delta_{\mathrm{light}}$ with a special boundary condition. Going to center of mass coordinates, we can write

$$\frac{1}{2M}\Delta_{\mathrm{heavy}} + \frac{1}{2m}\Delta_{\mathrm{light}} = \frac{1}{2(m+M)}\Delta_{\mathrm{cm}} + \frac{M+m}{2mM}\Delta_{\mathrm{dis}},$$

so that the special boundary condition will be placed on the displacement coordinate corresponding to Δ_{dis} and follows in analogy with that a single particle in a point potential as discussed