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Scaling properties of functionals and existence of constrained minimizers

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Abstract

In this paper we develop a new method to prove the existence of minimizers for a class of constrained minimization problems on Hilbert spaces that are invariant under translations. Our method permits to exclude the dichotomy of the minimizing sequences for a large class of functionals. We introduce family of maps, called scaling paths, that permits to show the strong subadditivity inequality. As byproduct the strong convergence of the minimizing sequences (up to translations) is proved. We give an application to the energy functional I associated to the Schrödinger–Poisson equation in $\mathbb{R}^3$

$$i\psi_t + \Delta\psi - \big(|x|^{-1} * |\psi|^2\big)\psi + |\psi|^{p-2}\psi = 0$$

when $2 < p < 3$. In particular we prove that I achieves its minimum on the constraint $\{u \in H^1(\mathbb{R}^3): \|u\|_2 = \rho\}$ for every sufficiently small $\rho > 0$. In this way we recover the case studied in Sanchez and Soler (2004) [20] for $p = 8/3$ and we complete the case studied by the authors for $3 < p < 10/3$ in Bellazzini and Siciliano (2011) [4].

Keywords: Constrained minimization; Subadditivity inequality; Schrödinger–Poisson equations; Standing waves

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1. Introduction

The existence of minimizers for constrained functionals is an interesting problem either from a mathematical or from a physical point of view. Indeed in many applications often appears a C^1 functional whose critical points restricted to some constraint have a relevant physical meaning. For example, in Schrödiger-type equations the existence of standing wave solutions can be proved by finding minimizers for the energy functional on L^2 constraint.

In this paper, having in mind an application to a Schrödinger–Poisson equation, we study the existence of minimizers for a class of functionals defined on a Hilbert space.

We consider $\mathcal{H}$, $\mathcal{H}_1$ two Hilbert spaces of functions defined in $\mathbb{R}^N$, with norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$ satisfying

$$\big\|u(\cdot+a)\big\|_{\mathcal{H}}=\big\|u(\cdot)\big\|_{\mathcal{H}},\qquad \big\|u(\cdot+a)\big\|_{\mathcal{H}_1}=\big\|u(\cdot)\big\|_{\mathcal{H}_1}\quad \text{for all } a\in\mathbb{R}^N.$$

Assume that $\mathcal{H}\subset\mathcal{H}_1$, $\mathcal{H}\subset L^2(\mathbb{R}^N)$ with

$$c_1\big(\|\cdot\|^2_{\mathcal{H}_1}+\|\cdot\|^2_{L^2(\mathbb{R}^N)}\big)\leqslant\|\cdot\|^2_{\mathcal{H}}\leqslant c_2\big(\|\cdot\|^2_{\mathcal{H}_1}+\|\cdot\|^2_{L^2(\mathbb{R}^N)}\big)$$

where $L^2(\mathbb{R}^N)$ is the usual Lebesgue space. Let $I:\mathcal{H}\to\mathbb{R}$ be a functional of the following form

$$I(u):=\frac{1}{2}\|u\|^2_{\mathcal{H}_1}+T(u) \tag{1.1}$$

where the nonlinear operator $T\in C^1(\mathcal{H},\mathbb{R})$ satisfies some suitable assumptions. In particular we require that T is invariant for the noncompact group of translations in $\mathbb{R}^N$ so that also the functional I is "translation invariant", i.e. it satisfies $I(u(x+a))=I(u(x))$.

We look at the constrained minimization problem

$$I_{\rho^2}:=\inf_{B_\rho}I(u)\quad(\text{we agree } I_0=0) \tag{1.2}$$

where $B_\rho=\{u\in\mathcal{H}\colon\ \|u\|_2=\rho\}$ and $I_{\rho^2}>-\infty$ is assumed.

The main difficulty for translation invariant functionals is due to the lack of compactness of the (bounded) minimizing sequences $\{u_n\}\subset B_\rho$; indeed the minimizing sequence $\{u_n\}$ could run off to spatial infinity and/or spread uniformly in space. So even up to translations two possible bad scenarios are possible:

- (vanishing) $u_n\rightharpoonup 0$;
- (dichotomy) $u_n\rightharpoonup\bar u\neq 0$ and $0<\|\bar u\|_2<\rho$.

The general strategy in the applications (see for instance [13] in case of weakly lower semicontinuous functionals) is to prove that any minimizing sequence weakly converges, up to translation, to a function $\bar u$ which is different from zero, excluding the vanishing case. Then one has to show that $\|\bar u\|_2=\rho$, which proves that dichotomy does not occur.