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Journal of Functional Analysis 258 (2010) 161–195

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Stability estimate for an inverse problem for the magnetic Schrödinger equation from the Dirichlet-to-Neumann map

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Received 10 March 2009; accepted 17 June 2009

Available online 4 July 2009

Communicated by J. Coron

Abstract

We consider the problem of stability estimate of the inverse problem of determining the magnetic field entering the magnetic Schrödinger equation in a bounded smooth domain of $\mathbb{R}^n$ with input Dirichlet data, from measured Neumann boundary observations. This information is enclosed in the dynamical Dirichlet-to-Neumann map associated to the solutions of the magnetic Schrödinger equation. We prove in dimension $n \geqslant 2$ that the knowledge of the Dirichlet-to-Neumann map for the magnetic Schrödinger equation measured on the boundary determines uniquely the magnetic field and we prove a Hölder-type stability in determining the magnetic field induced by the magnetic potential.

Keywords: Stability estimate; Schrödinger inverse problem; Magnetic field; Dirichlet-to-Neumann map

1. Introduction

In this paper we study an inverse problem for the dynamical Schrödinger equation in the presence of a magnetic potential. Such an equation appears naturally in some mathematical models related to certain quantum dynamical systems. We shall consider the physically important case of a real valued magnetic potential. We will see below that in this case one has conservation of

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.06.010

charge. The dynamical Schrödinger equation plays also an important role in geometry. We refer to [42] and references therein for more details.

Throughout this paper we assume that Ω is an open bounded subset of $\mathbb{R}^n$, $n \geqslant 2$, with $\mathcal{C}^\infty$ boundary Γ. Given $T > 0$, we consider the following initial boundary value problem (IBVP in short) for the Schrödinger equation with a magnetic potential, where $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$,

$$\begin{cases} (i\partial_t + \Delta_A)u = 0, & \text{in } Q, \\ u(0,\cdot) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where

$$\Delta_A = \sum_{j=1}^{n} (\partial_j + i a_j)^2 = \Delta + 2iA \cdot \nabla + i\,\mathrm{div}(A) - |A|^2.$$

Here $A = (a_j)_{1 \leqslant j \leqslant n} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ is the magnetic potential. We may define the operator

$$\Lambda_A(f) = (\partial_\nu + iA \cdot \nu)u, \quad f \in L^2(\Sigma),$$

where $\nu = \nu(x)$ denotes the unit outward normal to Γ at x. We call Λ_A the Dirichlet-to-Neumann map (DN map in short) associated to the IBVP (1.1).

We consider the inverse problem to know whether the DN map Λ_A determines uniquely the magnetic potential A.

First of all, let us observe that there is an obstruction to uniqueness. In fact as it was noted in [19], the DN map is invariant under the gauge transformation of the magnetic potential. Namely, given $\Psi \in \mathcal{C}^1(\overline{\Omega})$ such that $\Psi|_\Gamma = 0$ one has

$$e^{-i\Psi} \Delta_A e^{i\Psi} = \Delta_{A+\nabla\Psi}, \qquad e^{-i\Psi} \Lambda_A e^{i\Psi} = \Lambda_{A+\nabla\Psi}, \tag{1.2}$$

and $\Lambda_A = \Lambda_{A+\nabla\Psi}$. Therefore, the magnetic potential A cannot be uniquely determined by the DN map Λ_A. From a geometric view point this can be seen as follows. The vector field A defines the connection given by the one form $\alpha_A = \sum_{j=1}^n a_j\,dx_j$, and the non-uniqueness manifested in (1.2) says that the best we could hope to reconstruct from the DN map Λ_A is the 2-form called the magnetic field $d\alpha_A$ given by

$$d\alpha_A = \sum_{i,j=1}^{n} \left(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} \right) dx_j \wedge dx_i.$$

Physically, our inverse problem consists in determining the magnetic field $d\alpha_A$ induced by the magnetic potential A of an inhomogeneous medium by probing it with disturbances generated on the boundary. The data are responses of the medium to these disturbances which are measured on the boundary and the goal is to recover the magnetic field $d\alpha_A$ which describes the property of the medium. Here we assume that the medium is quiet initially and f is a disturbance which is used to probe the medium. Roughly speaking, the data is $(\partial_\nu + i\nu \cdot A)u$ measured on the boundary for different choices of f.