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Hardy type inequality and application to the stability of degenerate stationary waves

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Abstract

This paper is concerned with the asymptotic stability of degenerate stationary waves for viscous conservation laws in the half space. It is proved that the solution converges to the corresponding degenerate stationary wave at the rate $t^{-\alpha/4}$ as $t \to \infty$, provided that the initial perturbation is in the weighted space $L^2_\alpha = L^2(\mathbb{R}_+; (1+x)^\alpha)$ for $\alpha < \alpha_c(q) := 3 + 2/q$, where q is the degeneracy exponent. This restriction on α is best possible in the sense that the corresponding linearized operator cannot be dissipative in L^2_α for $\alpha > \alpha_c(q)$. Our stability analysis is based on the space-time weighted energy method combined with a Hardy type inequality with the best possible constant.

Keywords: Viscous conservation laws; Degenerate stationary waves; Asymptotic stability; Hardy inequality

1. Introduction

We study the stability problem of degenerate stationary waves for viscous conservation laws in the half space $x > 0$:

$$
\begin{aligned}
&u_t + f(u)_x = u_{xx}, \\
&u(0,t) = -1, \qquad u(x,0) = u_0(x).
\end{aligned}
\tag{1.1}
$$

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Here the initial function is assumed to satisfy $u_0(x) \to 0$ as $x \to \infty$, and $f(u)$ is a smooth function of the form

$$f(u) = \frac{1}{q}(-u)^{q+1}\bigl(1 + g(u)\bigr), \qquad f''(u) > 0 \quad \text{for } -1 \leqslant u < 0, \tag{1.2}$$

where q is a positive integer (degeneracy exponent) and $g(u) = O(|u|)$ for $u \to 0$. Since $f(0) = f'(0) = 0$ and $f(u)$ is strictly convex for $-1 \leqslant u < 0$, we see that $f(u) > 0$ for $-1 \leqslant u < 0$. In particular, we have $1 + g(u) > 0$ for $-1 \leqslant u \leqslant 0$. In this situation, the corresponding stationary problem admits a unique solution $\phi(x)$ (called *degenerate stationary wave*), which verifies

$$\begin{aligned} &\phi_x = f(\phi), \\ &\phi(0) = -1, \qquad \phi(x) \to 0 \quad \text{as } x \to \infty. \end{aligned} \tag{1.3}$$

We see easily that $\phi(x)$ behaves like $\phi(x) \sim -(1+x)^{-1/q}$. In particular, we have $\phi(x) = -(1+x)^{-1/q}$ when $g(u) \equiv 0$.

To discuss the stability of the degenerate stationary wave $\phi(x)$, we introduce the perturbation v by $u(x,t) = \phi(x) + v(x,t)$ and rewrite the problem (1.1) as

$$\begin{aligned} &v_t + \bigl(f(\phi + v) - f(\phi)\bigr)_x = v_{xx}, \\ &v(0,t) = 0, \qquad v(x,0) = v_0(x), \end{aligned} \tag{1.4}$$

where $v_0(x) = u_0(x) - \phi(x)$, and $v_0(x) \to 0$ as $x \to \infty$. The stability of degenerate stationary waves was first studied in [15]. It was proved in [15] that if the initial perturbation $v_0(x)$ is in the weighted space L^2_α, then the perturbation $v(x,t)$ decays in L^2 at the rate $t^{-\alpha/4}$ as $t \to \infty$, provided that $\alpha < \alpha_*(q)$, where

$$\alpha_*(q) := \bigl(q + 1 + \sqrt{3q^2 + 4q + 1}\bigr)/q.$$

The decay rate $t^{-\alpha/4}$ obtained in [15] would be optimal but the restriction $\alpha < \alpha_*(q)$ was not very sharp. The main purpose of this paper is to relax this restriction. Indeed, by employing the space–time weighted energy method in [15] and by applying a Hardy type inequality with the best possible constant (see Proposition 2.3), we show the same decay rate $t^{-\alpha/4}$ under the weaker restriction $\alpha < \alpha_c(q) := 3 + 2/q$ (see Theorem 4.1). Notice that $\alpha_*(q) < \alpha_c(q)$. It is interesting to note that a similar restriction on the weight is imposed also for the stability of degenerate shock profiles (see [10]). We remark that our stability result for degenerate stationary waves is completely different from those for non-degenerate case. In fact, for non-degenerate stationary waves, we have the better decay rate $t^{-\alpha/2}$ for the perturbation without any restriction on α. See [4–6,14,16] for the details. See also [2,7,9,11] for the related stability results for stationary waves.

In this paper we also discuss the dissipativity of the following linearized operator associated with (1.4):

$$Lv = v_{xx} - \bigl(f'(\phi)v\bigr)_x. \tag{1.5}$$