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Uniform K-homology theory

Ján Špakula

Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

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Abstract

We define a uniform version of analytic K-homology theory for separable, proper metric spaces. Furthermore, we define an index map from this theory into the K-theory of uniform Roe C*-algebras, analogous to the coarse assembly map from analytic K-homology into the K-theory of Roe C*-algebras. We show that our theory has a Mayer–Vietoris sequence. We prove that for a torsion-free countable discrete group Γ, the direct limit of the uniform K-homology of the Rips complexes of Γ, $\lim_{d\to\infty} K_*^u(P_d\Gamma)$, is isomorphic to $K_*^{\mathrm{top}}(\Gamma, \ell^\infty\Gamma)$, the left-hand side of the Baum–Connes conjecture with coefficients in $\ell^\infty\Gamma$. In particular, this provides a computation of the uniform K-homology groups for some torsion-free groups. As an application of uniform K-homology, we prove a criterion for amenability in terms of vanishing of a "fundamental class", in spirit of similar criteria in uniformly finite homology and K-theory of uniform Roe algebras.

MSC: primary 46L80

Keywords: Analytic K-homology; Coarse assembly map; Uniform Roe algebra

1. Introduction

The analytic K-homology theory of a second countable locally compact Hausdorff topological space X (see e.g. [8]) can be understood as an attempt to organize the elliptic differential operators over the space X into an abelian group. The (higher) indices of these operators can be interpreted as K-theory elements over C^*X, the Roe C*-algebra [12]. The Coarse Baum–Connes

E-mail address: jan.spakula@uni-muenster.de.

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and coarse Novikov conjectures assert certain properties of this index (or coarse assembly) map $\mu : K_*(X) \to K_*(C^*X)$, and have applications in geometry (see e.g. [12,15]). Also, the Coarse Baum–Connes conjecture can be viewed as an algorithm to compute the K-theory of Roe C*-algebras. In this spirit, the work presented here is setting up a framework for obtaining an algorithm to compute the K-theory groups of uniform Roe C*-algebras.

In this paper, we define a refined version of analytic K-homology theory. We loosely follow the exposition [8] of analytic K-homology. The main idea is to quantify "how well approximable by finite rank operators" are various compact operators appearing in the definition of a Fredholm module.

Our theory, compared to the classical K-homology, has some advantages (the theory becomes sensitive to some coarse properties, for instance amenability), but also some disadvantages (the K-theory of uniform Roe algebras tends to be uncountable if nonzero).

The theory exhibits similarities to the uniformly finite homology theory of Block and Weinberger [3,4], which should be connected to our theory via a Chern character map. This is analogous to the Chern map from analytic K-homology into the locally finite homology groups.

Using estimates from [11], we show that some elliptic operators coming from geometry give rise to uniform K-homology classes. Furthermore, we construct an index map μ_u from uniform K-homology into the K-theory of uniform Roe C*-algebras. The original example of a coarse index theorem [11] is actually carried out in this uniform context.

We prove that amenability of a metric space is equivalent to non-vanishing of a "fundamental class" in the uniform K-homology of the space. Our criterion is parallel to similar criteria in the uniformly finite homology [3] and K-theory of uniform Roe algebras [6]. Our proof borrows ideas from both of these papers.

In the case when the space in question is a Cayley graph of a countable torsion–free group Γ, we show that $\lim_{d\to\infty} K_*^u(P_d\Gamma)$, the direct limit of uniform K-homologies of its Rips complexes is naturally isomorphic to $K_*^{\mathrm{top}}(\Gamma, \ell^\infty\Gamma)$, the left-hand side of the Baum–Connes conjecture for the group Γ with coefficients in $\ell^\infty(\Gamma)$. This is analogous to a result of Yu [14], where he shows the equivalence of the Coarse Baum–Connes conjecture and the Baum–Connes conjecture with coefficients in $\ell^\infty(\Gamma, \mathcal{K})$. This statement is true without any assumption on torsion; it is open whether the torsion-free assumption can be dropped in the uniform setting. On the other hand, since the Baum–Connes conjecture with commutative coefficients is known for a number of torsion–free groups, this result provides a computation of some uniform K-homology groups.

The structure of this paper is as follows: Section 2 introduces the uniform K-homology groups, and in Section 3 we prove that certain Dirac-type differential operators give rise to uniform K-homology classes. Sections 4 and 5 are devoted to proving the Mayer–Vietoris sequence in our theory. We turn to coarse geometry and the index map in Sections 7–9. The connection between the uniform K-homology of a group Γ and the Baum–Connes conjecture with coefficients in $\ell^\infty\Gamma$ is shown in Section 10. In the final Section 11, we prove our criterion for amenability.

2. Uniform K-homology groups

In this paper, the spaces are separable proper metric spaces, unless explicitly specified otherwise. Throughout the paper, X shall stand for such a space, and d will denote its metric. Finally, to avoid set-theoretic difficulties, all Hilbert spaces are assumed to be separable.

Recall the definition of Fredholm modules—the representatives of cycles in the classical analytic K-homology theory.