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Isomorphism of Hilbert modules over stably finite C*-algebras

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Abstract

It is shown that if A is a stably finite C*-algebra and E is a countably generated Hilbert A-module, then E gives rise to a compact element of the Cuntz semigroup if and only if E is algebraically finitely generated and projective. It follows that if E and F are equivalent in the sense of Coward, Elliott and Ivanescu (CEI) and E is algebraically finitely generated and projective, then E and F are isomorphic. In contrast to this, we exhibit two CEI-equivalent Hilbert modules over a stably finite C*-algebra that are not isomorphic.

Keywords: C*-algebras; Hilbert modules; Cuntz semigroup; compact

1. Introduction

In [3] a new equivalence relation—we will call it *CEI equivalence*—on Hilbert modules was introduced. In general CEI equivalence is weaker than isomorphism, but it was shown that if A has stable rank one, then it is the same as isomorphism [3, Theorem 3]. Quite naturally, the authors wondered whether their result could be extended to the stably finite case. Unfortunately, it cannot. In Section 4, we give examples of Hilbert modules over a stably finite C*-algebra which are CEI-equivalent, but not isomorphic. On the other hand, we show in Section 3 that

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CEI equivalence amounts to isomorphism when restricted to "compact" elements of the Cuntz semigroup, in the stably finite case.

2. Definitions and preliminaries

Throughout this note all C*-algebras are assumed to be separable and all Hilbert modules are assumed to be right modules and countably generated. We will follow standard terminology and notation in the theory of Hilbert modules (see, for example, [5]). In particular, $\mathcal{K}$ denotes the compact operators on $\ell^2(\mathbb{N})$, while $\mathcal{K}(E)$ will denote the "compact" operators on a Hilbert module E.

For the reader's convenience, we recall a few definitions that are scattered throughout [3].

Definition 2.1. If $E \subset F$ are Hilbert A-modules, we say E is *compactly contained in* F if there exists a self-adjoint $T \in \mathcal{K}(F)$ such that $T|_E = \mathrm{id}_E$. In this situation we write $E \subset\subset F$.

Note that $E \subset\subset E$ if and only if $\mathcal{K}(E)$ is unital; it can be shown that this is also equivalent to E being algebraically finitely generated and projective (in the purely algebraic category of right A-modules)—see the proof of [3, Corollary 5] (this part of the proof did not require the assumption of stable rank one).

Definition 2.2. We say a Hilbert A-module E is *CEI subequivalent* to another Hilbert A-module F if every compactly contained submodule of E is isomorphic to a compactly contained submodule of F.

We say E and F are *CEI equivalent* if they are CEI subequivalent to each other—i.e., a third Hilbert A-module X is isomorphic to a compactly contained submodule of E if and only if X is isomorphic to a compactly contained submodule of F.

Definition 2.3. We let $Cu(A)$ denote the set of Hilbert A-modules, modulo CEI equivalence. The class of a module E in $Cu(A)$ will be denoted $[E]$.

It turns out that $Cu(A)$ is an abelian semigroup with $[E] + [F] := [E \oplus F]$. (Note: it is not even obvious that this is well defined!) Moreover $Cu(A)$ is partially ordered—$[E] \leqslant [F] \iff E$ is CEI subequivalent to F—and every increasing sequence has a supremum (i.e., least upper bound). See [3, Theorem 1] for proofs of these facts.

Definition 2.4. An element $x \in Cu(A)$ is *compact* (in the order-theoretic sense) if for every increasing sequence $\{x_n\} \subset Cu(A)$ with $x \leqslant \sup_n x_n$ there exists $n_0 \in \mathbb{N}$ such that $x \leqslant x_{n_0}$.

For a unital C*-algebra A, *stable finiteness* means that for every $n \in \mathbb{N}$, $M_n(A)$ contains no infinite projections. In the nonunital case there are competing definitions, but it seems most popular to say A is stably finite if the unitization $\tilde{A}$ is stably finite, so this is the definition we will use.

3. Main results

The proof of our first lemma is essentially contained in the proof of [3, Corollary 5].