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Instability of nonlinear dispersive solitary waves

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Abstract

We consider linear instability of solitary waves of several classes of dispersive long wave models. They include generalizations of KDV, BBM, regularized Boussinesq equations, with general dispersive operators and nonlinear terms. We obtain criteria for the existence of exponentially growing solutions to the linearized problem. The novelty is that we dealt with models with nonlocal dispersive terms, for which the spectra problem is out of reach by the Evans function technique. For the proof, we reduce the linearized problem to study a family of nonlocal operators, which are closely related to properties of solitary waves. A continuation argument with a moving kernel formula is used to find the instability criteria. These techniques have also been extended to study instability of periodic waves and of the full water wave problem.

Keywords: Instability; Solitary waves; Dispersive long waves

1. Introduction

We consider the stability and instability of solitary wave solutions of several classes of equations modeling weakly nonlinear, dispersive long waves. More specifically, we establish criteria for the linear exponential instability of solitary waves of BBM, KDV, and regularized Boussinesq type equations. These equations respectively have the forms:

1. BBM type

$$\partial_t u + \partial_x u + \partial_x f(u) + \partial_t \mathcal{M} u = 0; \tag{1.1}$$

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2. KDV type

$$\partial_t u + \partial_x f(u) - \partial_x \mathcal{M}u = 0; \tag{1.2}$$

3. Regularized Boussinesq (RBou) type

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2 f(u) + \partial_t^2 \mathcal{M}u = 0. \tag{1.3}$$

Here, the pseudo-differential operator $\mathcal{M}$ is defined as

$$(\mathcal{M}g)^{\wedge}(k) = \alpha(k)\hat{g}(k),$$

where $\hat{g}$ is the Fourier transformation of g. Throughout this paper, we assume:

(i) f is C^1 with $f(0) = f'(0) = 0$, and $f(u)/u \to \infty$.
(ii) $a|k|^m \leqslant \alpha(k) \leqslant b|k|^m$ for large k, where $m \geqslant 1$ and $a, b > 0$.

If $f(u) = u^2$ and $\mathcal{M} = -\partial_x^2$, the above equations recover the original BBM [11], KDV [28], and regularized Boussinesq [49] equations, which have been used to model the unidirectional propagation of water waves of long wavelengths and small amplitude. As explained in [11], the nonlinear term $f(u)$ is related to nonlinear effects suffered by the waves being modeled, while the form of the symbol α is related to dispersive and possibly, dissipative effects. If $\alpha(k)$ is a polynomial function of k, then $\mathcal{M}$ is a differential operator and in particular is a local operator. On the other hand, in many situations in fluid dynamics and mathematical physics, equations of the above types arise in which $\alpha(k)$ is not a polynomial and hence the operator $\mathcal{M}$ is non-local. Some examples include: Benjamin–Ono equation [8], Smith equation [45] and intermediate long-wave equation [29], which are of KDV types with $\alpha(k) = |k|$, $\sqrt{1+k^2} - 1$ and $k\coth(kH) - H^{-1}$, respectively.

Below we assume $\alpha(k) \geqslant 0$, since the results and proofs can be easily modified for cases of sign-changing symbols (see Section 5, (b)). Each of Eqs. (1.1)–(1.3) admits solitary-wave solutions of the form $u(x,t) = u_c(x - ct)$ for $c > 1$, $c > 0$ and $c^2 > 1$, respectively, where $u_c(x) \to 0$ as $|x| \to \infty$. For example, the KDV solitary-wave solutions have the form [28]

$$u_c(x) = 3c\,\mathrm{sech}^2(\sqrt{c}x/2)$$

and for the Benjamin–Ono equation [8]

$$u_c(x) = \frac{4c}{1 + c^2x^2}.$$

For a broad class of symbols α, the existence of solitary-wave solutions has been established [10,12]. For many equations such as the classical KDV and BBM equations, the solitary waves are positive, symmetric and single-humped. But the oscillatory solitary waves with multiple humps are not uncommon [5,7], especially for the sign changing $\alpha(k)$. In our study, we do not assume any additional property of solitary waves besides their decay at infinity. We consider the linearized equations around solitary waves in the traveling frame $(x - ct, t)$ and seek a growing mode solution of the form $e^{\lambda t}u(x)$ with $\mathrm{Re}\,\lambda > 0$. Define the operator $\mathcal{L}_0$ by (2.2), (4.4), and