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Induced and coinduced Banach Lie–Poisson spaces and integrability

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Abstract

The Poisson induction and coinduction procedures are used to construct Banach Lie–Poisson spaces as well as related systems of integrals in involution. This general method applied to the Banach Lie–Poisson space of trace class operators leads to infinite Hamiltonian systems of k-diagonal trace class operators which have infinitely many integrals. The bidiagonal case is investigated in detail.

Keywords: Banach Lie–Poisson space; Induction; Coinduction; Momentum map; Coadjoint orbit; Integrable Hamiltonian system

Contents

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1. Introduction

This paper continues the investigation of Banach Lie–Poisson spaces introduced in [12] and also studied in [4,13]. The theory of Banach Lie–Poisson spaces gives a natural generalization in the functional analytical context of the Poisson geometry of finite-dimensional integrable Hamiltonian systems. It gives also a solid mathematical foundation for the theory of Hamiltonian systems with infinitely many degrees of freedom. The interest in these system was initiated in [3] and [8] and from then on they have played an important role in mathematics and physics. With few notable exceptions, for infinite-dimensional systems, the Lie–Poisson bracket formulation is mostly formal. It is our belief that these formal approaches can be given a solid functional analytic underpinning. The present paper formulates such an approach for various families of integrable systems which arise in a natural way when one investigates Banach Lie–Poisson spaces of trace class operators.

The paper is organized as follows. Section 2 presents the general theory of induced and coinduced Banach Lie–Poisson structures (Propositions 2.1–2.4) and gives a method of construction of systems of integrals in involution. The associated involution theorem (Proposition 2.2 and Corollary 2.3) is an analogue of the classical R-matrix method for Banach Lie–Poisson spaces.

Section 3 investigates the Banach Lie–Poisson geometry of several classes of spaces of trace class operators. The general constructions of Section 2 are implemented explicitly to these spaces. The multi-diagonal Banach Lie group, its Lie algebra, and its dual are introduced and studied (Propositions 3.1 and 3.2). The naturally induced and coinduced Poisson structures on the preduals of their Banach Lie algebras are presented.

Section 4 formulates the equations of motion induced by the Casimir functions of the Banach Lie–Poisson space of trace class operators relative to the various induced and coinduced Poisson brackets discussed previously. These systems represent a k-diagonal version of the semi-infinite Toda system which is obtained from this point of view if $k = 2$. The solution of the systems associated to two different splittings of the space of trace class operators in terms of group decompositions are also presented.

Section 5 emphasizes the important particular case of bidiagonal operators. The Banach Lie group of upper bidiagonal bounded operators is studied in detail and the topological and symplectic structure of the generic coadjoint orbit is presented (Proposition 5.1). The Banach space analogue of the Flaschka map (defined for the first time in [7]) is analyzed and its relationship to the coadjoint orbits is pointed out (Propositions 5.2 and 5.3). There are new, typical infinite-dimensional, phenomena that appear in this context. For example, as opposed to the finite-dimensional case, the Banach space of lower bidiagonal trace class operators does not form a single coadjoint orbit and there are non-algebraic invariants for the coadjoint orbits. As an example of the theory, the semi-infinite Toda lattice is rigorously investigated using the method of orthogonal polynomials first introduced, to our knowledge, in [5]. The explicit solution of this system is obtained, both in action–angle as well as in the original variables, thereby extending the formulas in [10] from the finite to the semi-infinite Toda lattice.

Conventions. In this paper all Banach manifolds and Lie groups are real. The definition of the notion of a Banach Lie subgroup follows Bourbaki [6], that is, a subgroup H of a Banach Lie group G is necessarily a submanifold (and not just injectively immersed). In particular, Banach Lie subgroups are necessarily closed.