Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 255 (2008) 3125–3148

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Non-spectral problem for a class of planar self-affine measures

Jian-Lin Li

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, PR China

Received 21 February 2008; accepted 2 April 2008

Available online 8 May 2008

Communicated by L. Gross

Abstract

The self-affine measure $\mu_{M,D}$ corresponding to an expanding matrix $M \in M_n(\mathbb{R})$ and a finite subset $D \subset \mathbb{R}^n$ is supported on the attractor (or invariant set) of the iterated function system $\{\phi_d(x) = M^{-1}(x+d)\}_{d\in D}$. The spectral and non-spectral problems on $\mu_{M,D}$, including the spectrum-tiling problem implied in them, have received much attention in recent years. One of the non-spectral problem on $\mu_{M,D}$ is to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them. In the present paper we show that if $a, b, c \in \mathbb{Z}$, $|a| > 1$, $|c| > 1$ and $ac \in \mathbb{Z} \setminus (3\mathbb{Z})$,

$$M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

then there exist at most 3 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 3 is the best. This extends several known conclusions. The proof of such result depends on the characterization of the zero set of the Fourier transform $\hat{\mu}_{M,D}$, and provides a way of dealing with the non-spectral problem.

Keywords: Iterated function system; Self-affine measure; Orthogonal exponentials; Spectral measure

1. Introduction

Invariant measures, such as self-similar measures, have recently found wide use in the theory of fractals, in dynamics, in harmonic analysis and in quasicrystals (cf. [1,6]). A measure μ is self-

E-mail address: jllimath@yahoo.com.cn.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.04.001

similar if it is a convex combination of a given set S of transformations applied to the measure itself. In the literature, one usually restricts attention to the case where the set S is finite. Then, an iterated function system (IFS) results, and varying S yields a rich family of measures μ. To get a manageable problem, further restrictions are placed on the transformations from S. E.g., that they are contractive, and that they fall in a definite class, such as conformal maps (giving equilibrium measures on Julia sets), or affine mappings. Here the affine case is considered. Our IFS $\{\phi_d(x)\}_{d\in D}$ consists of the following affine maps on $\mathbb{R}^n$,

$$\phi_d(x) = M^{-1}(x+d) \quad \bigl(x \in \mathbb{R}^n\bigr),$$

where $M \in M_n(\mathbb{R})$ is an $n \times n$ expanding real matrix (that is, all the eigenvalues of the real matrix M have moduli > 1), and $D \subset \mathbb{R}^n$ is a finite subset of the cardinality $|D|$. We denote the corresponding measure by $\mu_{M,D}$, which is a unique probability measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|} \sum_{d\in D} \mu \circ \phi_d^{-1}. \tag{1.1}$$

Such a measure $\mu_{M,D}$ is supported on the attractor (or invariant set) $T(M,D)$ of the affine IFS $\{\phi_d(x)\}_{d\in D}$ (cf. [7,12]), and is called a *self-affine measure*.

Since this affine case includes restrictions of n-dimensional Lebesgue measure, Cantor measures, and IFS fractal measures, say on Sierpinski gaskets, it is natural to ask for Fourier duality. Can one get some kind of Fourier representation for $\mu_{M,D}$? We know from prior research on $L^2(\mu_{M,D})$ that a naive notion of orthogonal Fourier series is not feasible in general for affine IFSs. For example, the familiar middle 3rd Cantor set $T(M,D)$ corresponding to $M = 3$ and $D = \{0,2\}$, Jorgensen and Pedersen [18, Theorem 6.1] proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 2 elements. In the case when $M = p$, $p > 1$, is odd and $D = \{0,1\}$, Dutkay and Jorgensen [4, Theorem 5.1(i)] proved that there are no 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In this paper we will explore planar affine IFS-examples when the obstruction to getting a Fourier basis is extreme.

Recall that for a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a *spectral measure* if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that the exponential function system $E_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle}\colon \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$. The set Λ is then called a *spectrum* for μ; we also say that (μ,Λ) is a *spectral pair* (cf. [19]).

Spectral measure is a natural generalization of spectral set introduced by Fuglede [10] whose famous spectrum-tiling conjecture and its related problems have received much attention in recent years (cf. [6,24,25]). The spectral self-affine measure problem at the present day consists in determining conditions under which $\mu_{M,D}$ is a spectral measure, and has been studied in the papers [2–6,18,23,25,27,28,32] (see also [33,34] for the main goal). In the opposite direction, the non-spectral Lebesgue measure problem has been studied in the papers [10,11,15–17,22,26] and [13,14] where the conjecture that the disk has no more than 3 orthogonal exponentials is still unsolved. Correspondingly, the non-spectral problem on the self-affine measure consists of the following two classes:

(I) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (cf. [4,29]).