Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 256 (2009) 4030–4070
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Noncommutative hyperbolic geometry on the unit ball of $B(\mathcal{H})^n$ ☆

Gelu Popescu

Department of Mathematics, University of Texas at San Antonio, One UTSA Circle, San Antonio, TX 78249, USA

Received 3 October 2008; accepted 3 February 2009

Available online 20 February 2009

Communicated by N. Kalton

Abstract

In this paper we introduce a hyperbolic (Poincaré–Bergman type) distance δ on the noncommutative open ball

$$[B(\mathcal{H})^n]_1 := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n \colon \|X_1 X_1^* + \cdots + X_n X_n^*\|^{1/2} < 1\},$$

where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. It is proved that δ is invariant under the action of the free holomorphic automorphism group of $[B(\mathcal{H})^n]_1$, i.e.,

$$\delta(\Psi(X), \Psi(Y)) = \delta(X, Y), \quad X, Y \in [B(\mathcal{H})^n]_1,$$

for all $\Psi \in Aut([B(\mathcal{H})^n]_1)$. Moreover, we show that the δ-topology and the usual operator norm topology coincide on $[B(\mathcal{H})^n]_1$. While the open ball $[B(\mathcal{H})^n]_1$ is not a complete metric space with respect to the operator norm topology, we prove that $[B(\mathcal{H})^n]_1$ is a complete metric space with respect to the hyperbolic metric δ. We obtain an explicit formula for δ in terms of the reconstruction operator

$$R_X := X_1^* \otimes R_1 + \cdots + X_n^* \otimes R_n, \quad X := (X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1,$$

associated with the right creation operators $R_1, \ldots, R_n$ on the full Fock space with n generators. In the particular case when $\mathcal{H} = \mathbb{C}$, we show that the hyperbolic distance δ coincides with the Poincaré–Bergman

☆ Research supported in part by an NSF grant.

E-mail address: gelu.popescu@utsa.edu.

doi:10.1016/j.jfa.2009.02.003

distance on the open unit ball

$$\mathbb{B}_n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \|z\|_2 < 1\}.$$

We obtain a Schwarz–Pick lemma for free holomorphic functions on $[B(\mathcal{H})^n]_1$ with respect to the hyperbolic metric, i.e., if $F := (F_1, \ldots, F_m)$ is a contractive ($\|F\|_\infty \leqslant 1$) free holomorphic function, then

$$\delta\big(F(X), F(Y)\big) \leqslant \delta(X, Y), \quad X, Y \in \big[B(\mathcal{H})^n\big]_1.$$

As consequences, we show that the Carathéodory and the Kobayashi distances, with respect to δ, coincide with δ on $[B(\mathcal{H})^n]_1$. The results of this paper are presented in the more general context of Harnack parts of the closed ball $[B(\mathcal{H})^n]_1^-$, which are noncommutative analogues of the Gleason parts of the Gelfand spectrum of a function algebra.

Keywords: Noncommutative hyperbolic geometry; Noncommutative function theory; Poincaré–Bergman metric; Harnack part; Hyperbolic distance; Free holomorphic function; Free pluriharmonic function; Schwarz–Pick lemma

0. Introduction

Poincaré's discovery of a conformally invariant metric on the open unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ of the complex plane was a cornerstone in the development of complex function theory. The hyperbolic (Poincaré) distance is defined on $\mathbb{D}$ by

$$\delta_P(z, w) := \tanh^{-1}\left|\frac{z - w}{1 - \bar{z}w}\right|, \quad z, w \in \mathbb{D}.$$

Some of the basic and most important properties of the Poincaré distance are the following:

(1) the Poincaré distance is invariant under the conformal automorphisms of $\mathbb{D}$, i.e.,

$$\delta_P\big(\varphi(z), \varphi(w)\big) = \delta_P(z, w), \quad z, w \in \mathbb{D},$$

for all $\varphi \in Aut(\mathbb{D})$;

(2) the δ_P-topology induced on the open disc is the usual planar topology;

(3) $(\mathbb{D}, \delta_P)$ is a complete metric space;

(4) any analytic function $f : \mathbb{D} \to \mathbb{D}$ is distance-decreasing, i.e., satisfies

$$\delta_P\big(f(z), f(w)\big) \leqslant \delta_P(z, w), \quad z, w \in \mathbb{D}.$$

Bergman (see [2]) introduced an analogue of the Poincaré distance for the open unit ball of $\mathbb{C}^n$,

$$\mathbb{B}_n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \|z\|_2 < 1\},$$