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Rearrangement invariance of Rademacher multiplicator spaces

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Abstract

Let X be a rearrangement invariant function space on $[0,1]$. We consider the Rademacher multiplicator space $\Lambda(\mathcal{R},X)$ of all measurable functions x such that $x\cdot h\in X$ for every a.e. converging series $h=\sum a_n r_n\in X$, where (r_n) are the Rademacher functions. We study the situation when $\Lambda(\mathcal{R},X)$ is a rearrangement invariant space different from L^∞. Particular attention is given to the case when X is an interpolation space between the Lorentz space $\Lambda(\varphi)$ and the Marcinkiewicz space $M(\varphi)$. Consequences are derived regarding the behaviour of partial sums and tails of Rademacher series in function spaces.

Keywords: Rademacher functions; Rearrangement invariant spaces

Introduction

In this paper we study the behaviour of the Rademacher functions (r_n) in function spaces. Let $\mathcal{R}$ denote the set of all functions of the form $\sum a_n r_n$, where the series converges a.e. For a rearrangement invariant (r.i.) space X on $[0,1]$, let $\mathcal{R}(X)$ be the closed linear subspace of X given by $\mathcal{R}\cap X$. The *Rademacher multiplicator space* of X is the space $\Lambda(\mathcal{R},X)$ of all measurable functions $x:[0,1]\to\mathbb{R}$ such that $x\sum a_n r_n\in X$, for every $\sum a_n r_n\in\mathcal{R}(X)$. It is a

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Banach function space on [0, 1] when endowed with the norm

$$\|x\|_{\Lambda(\mathcal{R},X)} = \sup\left\{\left\|x\sum a_n r_n\right\|_X : \sum a_n r_n \in X,\ \left\|\sum a_n r_n\right\|_X \leqslant 1\right\}.$$

The space $\Lambda(\mathcal{R}, X)$ can be viewed as the space of operators from $\mathcal{R}(X)$ into the whole space X given by multiplication by a measurable function.

The Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ was firstly considered in [8] where it was shown that for a broad class of classical r.i. spaces X the space $\Lambda(\mathcal{R}, X)$ is not r.i. This result was extended in [2] to include all r.i. spaces such that the lower dilation index γ_{φ_X} of their fundamental function φ_X satisfies $\gamma_{\varphi_X} > 0$. This result motivated the study the symmetric kernel $\mathrm{Sym}(\mathcal{R}, X)$ of the space $\Lambda(\mathcal{R}, X)$, that is, the largest r.i. space embedded into $\Lambda(\mathcal{R}, X)$. The space $\mathrm{Sym}(\mathcal{R}, X)$ was studied in [2], where it was shown that, if X is an r.i. space satisfying the Fatou property and $X \supset L_N$, where L_N is the Orlicz space with $N(t) = \exp(t^2) - 1$, then $\mathrm{Sym}(\mathcal{R}, X)$ is the r.i. space with the norm $\|x\| := \|x^*(t)\log^{1/2}(2/t)\|_X$. It was also shown that any space X which has the Fatou property and is an interpolation space for the couple $(L\log^{1/2} L, L^\infty)$ can be realized as the symmetric kernel of a certain r.i. space. The opposite situation is when the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ is r.i. The simplest case of this situation is when $\Lambda(\mathcal{R}, X) = L^\infty$. In [1] it was shown that $\Lambda(\mathcal{R}, X) = L^\infty$ holds for all r.i. spaces X which are interpolation spaces for the couple (L^∞, L_N). It was shown in [3] that $\Lambda(\mathcal{R}, X) = L^\infty$ if and only if the function $\log^{1/2}(2/t)$ does not belong to the closure of L^∞ in X.

In this paper we investigate the case when the Rademacher multiplicator space $\Lambda(\mathcal{R}, X)$ is an r.i. space different from L^∞. Examples of this situation were considered in [2,8,9]. In all cases they were spaces X consisting of functions with exponential growth.

The paper is organized as follows. Section 1 is devoted to the preliminaries. In Section 2 we study technical conditions on an r.i. space X and its fundamental function φ. In Section 3 we present a sufficient condition for $\Lambda(\mathcal{R}, X)$ being r.i. (Theorem 3.4). For this, two results are needed. Firstly, that the symmetric kernel $\mathrm{Sym}(\mathcal{R}, X)$ is a maximal space (Proposition 3.1), and secondly, a condition, of independent interest, on the behaviour of logarithmic functions on an r.i. space (Proposition 3.3). Section 4 is devoted to the study of necessary conditions for $\Lambda(\mathcal{R}, X)$ being an r.i. space. This is done by separately studying conditions on partial sums and tails of Rademacher series (Propositions 4.1 and 4.2). Theorem 4.4 addresses the case when X in an interpolation space for the couple $(\Lambda(\varphi), M(\varphi))$, where $\Lambda(\varphi)$ and $M(\varphi)$ are, respectively, the Lorentz and Marcinkiewicz spaces with the fundamental function φ. Theorem 4.5 specializes the previous result for the case of $X = M(\varphi)$. We end presenting, in Section 5, examples which highlight certain features of the previous results.

1. Preliminaries

Throughout the paper a rearrangement invariant (r.i.) space X is a Banach space of classes of measurable functions on [0, 1] such that if $y^* \leqslant x^*$ and $x \in X$ then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$. Here x^* is the decreasing rearrangement of x, that is, the right continuous inverse of its distribution function: $n_x(\tau) = \lambda\{t \in [0, 1]\colon |x(t)| > \tau\}$, where λ is the Lebesgue measure on [0, 1]. Functions x and y are said to be equimeasurable if $n_x(\tau) = n_y(\tau)$, for all $\tau > 0$. The associated space (or Köthe dual) of X is the space X' of all functions y such that $\int_0^1 |x(t)y(t)|\,dt < \infty$, for every $x \in X$. It is an r.i. space. The space X' is a subspace of the topological dual X^*. If X' is a