Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 254 (2008) 1–83
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Compact quantum ergodic systems

Reiji Tomatsu

Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

Received 8 December 2005; accepted 21 August 2007

Available online 8 November 2007

Communicated by Alain Connes

Abstract

We develop theory of multiplicity maps for compact quantum groups. As an application, we obtain a complete classification of right coideal C^*-algebras of $C(SU_q(2))$ for $q \in [-1, 1) \setminus \{0\}$. They are labeled with Dynkin diagrams, but classification results for positive and negative cases of q are different. Many of the coideals are quantum spheres or quotient spaces by quantum subgroups, but we do have other ones in our classification list.

Keywords: Compact quantum groups; Ergodic actions; Right coideals

1. Introduction

The core of this paper consists of studying general compact quantum ergodic systems and classifying right coideals of $C(SU_q(2))$ for $q \in [-1, 1) \setminus \{0\}$. Our motivation relies on the works [20] and [21] by A. Wassermann, where he has established theory of multiplicity maps and classified ergodic systems of the compact group $SU(2)$. It is natural to ask whether his theory can be adapted to the classification of ergodic systems of a compact quantum group $SU_q(2)$, which is an example of a compact quantum group and we regard it as a deformed $SU(2)$ group with a parameter q. First of all, we look back upon ergodic actions of compact groups.

Ergodicity means that the fixed point algebra of given action becomes trivial. This strong condition derives several special properties. For example, the invariant state must be tracial and the multiplicities of irreducible representations are bounded by dimensions of their representation spaces [10]. In quantum setting, the multiplicities of them also become finite [4], however an

E-mail address: tomatsu@ms.u-tokyo.ac.jp.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.08.013

invariant state has no longer the tracial property and the multiplicities are bounded not by usual dimensions but by quantum dimensions which are larger than or equal to usual dimensions. Actually in [3], a great deal of examples of compact quantum ergodic systems which have the multiplicities strictly larger than their dimensions are constructed. Keeping in mind these phenomena, one shall notice there are rather differences between the ergodic actions of classical and quantum groups. In studying compact quantum ergodic systems, the main machineries treated in this paper are multiplicity maps and their diagrams developed by A. Wassermann [20]. Since multiplicity maps are defined via equivariant K-theory, one easily obtains its quantum version by using the work due to, for example, [1,4,19]. The most non-trivial important problem is whether there exists a common eigenvector of multiplicity maps or not. One shall notice that its existence is guaranteed by a tracial invariant state. For that problem, we show its existence in two cases, (1) A has a (not necessarily faithful) tracial state and $G = SU_q(2)$, (2) A is a right coideal.

With these results, we can proceed to classify all right coideals of $C(SU_q(2))$, which is the aim of the latter half of this paper. In the classical case, we know the continuous function algebra on the homogeneous space $H \setminus G$ by a closed subgroup $H \subset G$ gives a right coideal and its converse holds via the Gelfand–Naimark theorem on abelian C^*-algebras. However in the case of a quantum group, such a correspondence breaks down in general. For example, in [16], a one-parameter family of quantum spheres $C(S^2_{q,\lambda})$ is constructed, which consists of right coideals and most of them are not obtained by taking quotient by subgroups. Therefore, it is interesting to investigate the other non-quotient type right coideals. The most important information of a right coideal is its spectral pattern, that is, multiplicities of irreducible representations. We will see that multiplicity diagrams are labeled by closed subgroups of $SU(2)$ or $SU_{-1}(2)$ as McKay diagrams with respect to the fundamental two-dimensional representation and from those diagrams, one can compute the possible spectral patterns. These data enable us to classify right coideals into the several types by connected graphs of norm 2 as is used in the classification of ergodic systems of $SU(2)$. Then one can find absence in some spectral patterns. Their gaps often become a good obstruction for existence of ergodic systems. Then we carry out case-by-case study of them. In that procedure, one has to be careful of the essential effect of $|q| \neq 1$ and its sign. In fact, when we work on the negative q case, it is also needed to treat the graphs with a single loop at a vertex. As a result, right coideals of some types such as the regular polyhedrons do not appear in those cases. We state the main result on this classification.

(1) The case $0 < q < 1$. A right coideal must be one of type 1, $SU(2)$, $\mathbb{T}_n$, $\mathbb{T}$ and D^*_∞. When it is of type $\mathbb{T}$, then it is one of series of the quantum spheres. Otherwise it is uniquely determined by the type.
(2) The case $-1 < q < 0$. A right coideal must be one of type 1, $SU(2)$, $\mathbb{T}_n$, $\mathbb{T}$, D^*_∞ and D_1. When it is of type $\mathbb{T}$, then it is one of series of the quantum spheres. Otherwise it is uniquely determined by the type.
(3) The case $q = -1$. If a right coideal A is not of type $\mathbb{T}_n$ (odd $n \geqslant 3$) or D_n (odd $n \geqslant 1$), there exists a closed subgroup H in $SO_{-1}(3)$ such that A is $C(H \setminus SO_{-1}(3))$. If a right coideal A is of type $\mathbb{T}_n$ (odd $n \geqslant 3$), A is conjugated to $C(\mathbb{T}_n \setminus SU_{-1}(2))$ or $C^*(\eta^{\frac{n}{2}}, \hat{\eta}^{\frac{n}{2}})$. If a right coideal A is of type D_1, then A is conjugated to $C(D_1 \setminus SU_{-1}(2))$. If a right coideal A is of type D_n (odd $n \geqslant 3$), A is conjugated to $C(D_n \setminus SU_{-1}(2))$ or $C^*(\eta^{\frac{n}{2}})$. Here conjugation is given by the left action β^L_z of the maximal torus for some $z \in \mathbb{T}$.

In each of the above cases, uniqueness is up to conjugation by the left action of the maximal torus $\mathbb{T}$. On right coideals which are of type D^*_∞ in the case $0 < q < 1$, of types D^*_∞ and D_1