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ScienceDirect
Journal of Functional Analysis 260 (2011) 3222–3251
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Paley–Wiener spaces with vanishing conditions and Painlevé VI transcendents

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Received 17 September 2010; accepted 1 December 2010

Available online 9 December 2010

Communicated by Alain Connes

Abstract

We modify the classical Paley–Wiener spaces PW_x of entire functions of finite exponential type at most $x > 0$, which are square integrable on the real line, via the additional condition of vanishing at finitely many complex points $z_1, \ldots, z_n$. We compute the reproducing kernels and relate their variations with respect to x to a Krein differential system, whose coefficient (which we call the μ-function) and solutions have determinantal expressions. Arguments specific to the case where the "trivial zeros" $z_1, \ldots, z_n$ are in arithmetic progression on the imaginary axis allow us to establish for expressions arising in the theory a system of two non-linear first order differential equations. A computation, having this non-linear system at his start, obtains quasi-algebraic and among them rational Painlevé transcendents of the sixth kind as certain quotients of such μ-functions.

Keywords: Paley–Wiener; Painlevé; Krein

1. Introduction and summary of results

Let $\phi \in L^2(\mathbb{R}, dt)$ and $\mathcal{F}(\phi)(z) = \int_{-\infty}^{\infty} \phi(t)e^{izt}\,dt$ its Fourier transform. When ϕ is supported in $(-x, x)$, $f(z) = \mathcal{F}(\phi)(z)$ is an entire function of exponential type at most x. Conversely the Paley–Wiener theorem identifies the vector space PW_x of entire functions of exponential type at most x, square-integrable on the real line, as the Hilbert space of such Fourier transforms. Our convention for our scalar products is for them to be conjugate linear in the first factor and complex

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.12.002

linear in the second factor. Specifically $(\phi, \psi) = \int_{\mathbb{R}} \overline{\phi(t)}\psi(t)\,dt$, hence for the transforms f and g: $(f, g) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{f(z)}g(z)\,dz = (\phi, \psi)$.

The evaluator Z_z is the element of PW_x such that

$$\forall g \in PW_x, \quad g = \mathcal{F}(\psi), \qquad (Z_z, g) = g(z) = \int_{-x}^{x} e^{izt}\psi(t)\,dt = \left(\mathcal{F}\left(e^{-i\bar{z}t}\big|_{(-x,x)}\right), g\right). \tag{1}$$

Hence:

$$Z_z(w) = 2\frac{\sin((\bar{z} - w)x)}{\bar{z} - w} = \frac{e^{i\bar{z}x}e^{-iwx} - e^{-i\bar{z}x}e^{iwx}}{i(\bar{z} - w)}. \tag{2}$$

Let $E(w) = e^{-ixw}$ and $E^*(w) = \overline{E(\overline{w})}$. The evaluators in PW_x are given by

$$Z_z(w) = (Z_w, Z_z) = \frac{\overline{E(z)}E(w) - \overline{E^*(z)}E^*(w)}{i(\bar{z} - w)}. \tag{3}$$

Let us also define:

$$A(w) = \frac{1}{2}\big(E(w) + E^*(w)\big), \tag{4a}$$

$$B(w) = \frac{i}{2}\big(E(w) - E^*(w)\big). \tag{4b}$$

Then $E = A - iB$, $A = A^*$, $B = B^*$ and:

$$(Z_w, Z_z) = Z_z(w) = 2\frac{\overline{B(z)}A(w) - \overline{A(z)}B(w)}{\bar{z} - w}. \tag{5}$$

For the Paley–Wiener spaces, $A(w) = \cos(xw)$ is even and $B(w) = \sin(xw)$ is odd.

Let us consider generally a Hilbert space H, whose vectors are entire functions, and such that the evaluations at complex numbers are continuous linear forms, hence correspond to specific vectors Z_z. Let $\sigma = (z_1, \ldots, z_n)$ be a finite sequence of distinct complex numbers. We let H^σ be the closed subspace of H of functions vanishing at the z_i's. Let

$$\gamma(z) = \frac{1}{(z - z_1)\ldots(z - z_n)} \tag{6}$$

and define $H(\sigma) = \gamma(z)H^\sigma$:

$$H(\sigma) = \big\{F(z) = \gamma(z)f(z) \;\big|\; f \in H,\ f(z_1) = \cdots = f(z_n) = 0\big\}. \tag{7}$$

We introduced this notion in [5]. We say that $F(z) = \gamma(z)f(z)$ is the "complete" form of f, and refer to $z_1, \ldots, z_n$ as the "trivial zeros" of f. We give $H(\sigma)$ the Hilbert space structure which makes $f \mapsto F$ an isometry with H^σ. Let us note that evaluations $F \mapsto F(z)$ are again continuous