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A three-ball intersection property for u-ideals

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Abstract

First introduced by Casazza and Kalton, u-ideals are generalizations of M-ideals. We characterize u-ideals of Banach spaces using intersection properties of balls. We also give examples showing that our results are best possible.

Keywords: u-Ideals; Center of symmetry; Intersection properties of balls

1. Introduction

Let X be a closed subspace of a Banach space Y. In [4], Godefroy, Kalton and Saphar introduced the notion of an *ideal*. X is an ideal in Y if there exists a norm one projection P on Y^* with $\ker P = X^{\perp}$, the annihilator of X. According to Casazza and Kalton [2] X is a *u-ideal* in Y if $I - 2P$ is an isometry.

Godefroy, Kalton and Saphar studied u-ideals and related notions in [4]. Following [4] we introduce the following notation that will be used throughout. Let X be a closed subspace of a Banach space Y and let i_X be the natural embedding $i_X : X \to Y$. If P is a norm one projection on Y^* with $\ker P = X^{\perp}$ we may define a norm one operator $T : Y \to X^{**}$ by letting

$$\langle i_X^* y^*, T(y) \rangle = \langle y, P(y^*) \rangle \tag{1.1}$$

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for all $y \in Y$ and $y^* \in Y^*$. Then $T(x) = x$ for all $x \in X$ and if $I - 2P$ is an isometry then $\|y - 2i_X^{**}T(y)\| = \|y\|$ for all $y \in Y$. Furthermore, if we let $V = P(Y^*)$, then X being a u-ideal in Y means that $Y^* = V \oplus X^\perp$ and $\|v + \eta\| = \|v - \eta\|$ for all $v \in V$ and $\eta \in X^\perp$. X is said to be an *M-ideal* in Y [1,5] if this is an ℓ_1 sum, i.e. $Y^* = V \oplus_1 X^\perp$.

In this paper we will characterize u-ideals using intersection properties of balls. Characterizations of M-ideals by intersection properties of balls can be found already in Alfsen and Effros [1] where M-ideals were introduced (see e.g. [1, Theorems 5.8 and 5.9]).

In [7, Theorem 6.17] the second named author proved the following.

Theorem 1.1. *(See [7].) Let X be a closed subspace of a Banach space Y. The following statements are equivalent.*

(a) *X is an M-ideal in Y.*
(b) *For every $y \in B_Y$ the intersection $X \cap \bigcap_{i=1}^3 B(y + x_i, 1 + \varepsilon) \neq \emptyset$ for every collection of three points $(x_i)_{i=1}^3 \subset B_X$ and $\varepsilon > 0$.*

The following version of Lemma 3.3 in [4] motivates why we consider the type of balls we do in this paper.

Lemma 1.2. *(See [4].) Let X be a closed subspace of a Banach space Y. If X is a u-ideal in Y then for every $\varepsilon > 0$, $y \in Y$ and $x \in X$ there is an $x_0 \in X$ such that*

$$\|y + x - 2x_0\| < \|y - x\| + \varepsilon.$$

This inequality can be written $2x_0 \in B(y + x, \|y - x\| + \varepsilon)$. Using this we now state our first main result.

Theorem 1.3. *Let X be a closed subspace of a Banach space Y and let $y \in Y \setminus X$ and $Z =$ span$(X, \{y\})$. The following statements are equivalent.*

(a) *X is a u-ideal in Z.*
(b) $X^{\perp\perp} \cap \bigcap_{x \in X} B_{Z^{**}}(y + x, \|y - x\|) \neq \emptyset$.
(c) $X \cap \bigcap_{i=1}^n B_Z(y + x_i, \|y - x_i\| + \varepsilon) \neq \emptyset$ *for every finite collection $(x_i)_{i=1}^n \subset X$ and $\varepsilon > 0$.*
(d) $X \cap \bigcap_{i=1}^3 B_Z(y + x_i, \|y - x_i\| + \varepsilon) \neq \emptyset$ *for every collection of three points $(x_i)_{i=1}^3 \subset X$ and $\varepsilon > 0$.*

Theorem 1.3 will be proved in Section 2. That section also contains a general result, Proposition 2.6, about centers of symmetry for compact convex sets inspired by the proof of Theorem 1.3.

From Theorem 1.1 we see that X is an M-ideal in Y if and only if X is an M-ideal in Z for every subspace Z of Y containing X such that $\dim Z/X = 1$. It is also known (see e.g. [3, Théorème 2.14] or [8, Proposition 2.1]) that X is an ideal in Y if and only if X is an ideal in Z for every subspace Z of Y with $\dim Z/Y < \infty$; and this is *not* equivalent to X being an ideal in Z for every subspace Z of Y with $\dim Z/X = 1$ by an example of Lindenstrauss [9, p. 78]. For u-ideals we have the following.