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Homogeneous operators on Hilbert spaces of holomorphic functions ☆

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Abstract

In this paper we construct a large class of multiplication operators on reproducing kernel Hilbert spaces which are *homogeneous* with respect to the action of the Möbius group consisting of bi-holomorphic automorphisms of the unit disc $\mathbb{D}$. Indeed, this class consists of exactly those operators for which the associated unitary representation of the universal covering group of the Möbius group is multiplicity free. For every $m \in \mathbb{N}$ we have a family of operators depending on $m+1$ positive real parameters. The kernel function is calculated explicitly. It is proved that each of these operators is bounded, lies in the Cowen–Douglas class of $\mathbb{D}$ and is irreducible. These operators are shown to be mutually pairwise unitarily inequivalent.

Keywords: Homogeneous operators; Homogeneous holomorphic Hermitian vector boundle; Associated representation; Cowen–Douglas class; Reproducing kernel function

1. Introduction

A homogeneous operator on a Hilbert space $\mathcal{H}$ is a bounded operator T whose spectrum is contained in the closure of the unit disc $\mathbb{D}$ in $\mathbb{C}$ and is such that $g(T)$ is unitarily equivalent to T for all linear fractional transformations g which map $\mathbb{D}$ onto $\mathbb{D}$. This class of operators has been studied in a number of articles [1,3–7,10,17,19]. It is known that every irreducible homogeneous

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operator is a block shift, that is, $\mathcal{H}$ is the orthogonal direct sum of subspaces V_n, indexed by all integers, all non-negative integers or all non-positive integers, such that $T(V_n) \subseteq V_{n+1}$ for each n.

The case where $\dim V_n \leqslant 1$ for each n is completely known, the corresponding operators have been classified in [5]. The classification in the case where $\dim V_n \leqslant 2$ and T belongs to the Cowen–Douglas class of $\mathbb{D}$ is complete and the operators are explicitly described in [19]. Beyond this there are only some results of a general nature, and not too many examples are known (cf. [4]).

In the present article we construct a large family of examples. For every natural number m we construct a family depending on $m+1$ parameters. Each one of the examples is realized as the multiplication operator on a reproducing kernel space of vector-valued holomorphic functions. In the block shift realization, the reproducing kernel Hilbert space appears as $\bigoplus_{n\geqslant 0} V_n$ with $\dim V_n = n+1$ if $0 \leqslant n < m$ and $\dim V_n = m+1$ for $n \geqslant m$. The reproducing kernels are described explicitly. All our examples are irreducible operators and their adjoints belong to the Cowen–Douglas class.

For these results we have chosen a presentation as elementary as possible. This seemed to be appropriate since our goal was a complete explicit description of a class of fundamental examples. There exists, however, a deeper background from which our examples arise naturally. This is the subject of a short final section.

Here we show that the holomorphic Hermitian bundle corresponding to any homogeneous operator in the Cowen–Douglas class is homogeneous under the action of $\tilde{G}$, the universal covering group of the Möbius group G. This opens up a possibility of classifying all homogeneous Cowen–Douglas operators which we briefly sketch and which will be the subject of a subsequent article. It also shows that to every such operator there is an associated unitary representation of $\tilde{G}$. The operators constructed in the present article are those whose associated representation is multiplicity-free.

As suggested by the referee we explain here the connections of our results with the theory of Hilbert modules. In the notation of Section 2, let $\mathcal{M}$ be the Hilbert space $A^{(\alpha)}(\mathbb{D}) \otimes A^{(\beta)}(\mathbb{D})$ regarded as functions of two variables z, w, and let $\mathcal{M}_n$ be the subspace of functions vanishing to order n on the diagonal. $\mathcal{M}$ and $\mathcal{M}_n$ are Hilbert modules over the algebra $\mathcal{A}(\mathbb{D}^2)$ of continuous functions on the closure of $\mathbb{D}^2$ which are holomorphic on $\mathbb{D}^2$ (i.e. $\|f \cdot h\| \leqslant C\|f\|_\infty \|h\|$ for all $f \in \mathcal{A}(\mathbb{D}^2)$, $h \in A^{(\alpha)}(\mathbb{D}) \otimes A^{(\beta)}(\mathbb{D})$). So multiplication by z and w are bounded operators on $\mathcal{M}$ and they induce bounded operators, denoted M_1, M_2 on the quotient Hilbert module $\mathcal{M} \ominus \mathcal{M}_n$. They are clearly homogeneous under the representation induced by the natural representation of G on $A^{(\alpha)}(\mathbb{D}) \otimes A^{(\beta)}(\mathbb{D})$. It was observed already in [4] that all the homogeneous operators of [19] arise from this construction. It was not known before, but it follows from our results and from the remarks to be made below, that if we generalize the above construction by allowing arbitrary rescalings of the G-irreducible subspaces we can obtain a large family of inequivalent homogeneous operators depending on n parameters, namely the operators $M^{(\lambda,\boldsymbol{\mu})}$ of the present paper.

The way to handle the quotient module in [4,10,11,14,17] is to realize it as a space of holomorphic $\mathbb{C}^n$-valued functions on $\mathbb{D}$ under an isomorphism J_n defined by $(J_n h)(z)_j = \big(D^j h(z,w)\big)_{|z=w}$, where D is a differentiation transversal to the surface $z = w$ (the jet construction). Computations have been made with various choices of D. In [12] the "transvectants" of [16,17] are used, these have the advantage that they give the direct sum $A^{(\alpha+\beta)}(\mathbb{D}) + \cdots + A^{(\alpha+\beta+2n-2)}(\mathbb{D})$ as the image space, but M_1 gets mapped into a matricial operator (which the authors compute [13]). It is an essential observation of the referee that the spaces $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$ of