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Property A and CAT(0) cube complexes

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Abstract

Property A is a non-equivariant analogue of amenability defined for metric spaces. Euclidean spaces and trees are examples of spaces with Property A. Simultaneously generalising these facts, we show that finite-dimensional CAT(0) cube complexes have Property A. We do not assume that the complex is locally finite. We also prove that given a discrete group acting properly on a finite-dimensional CAT(0) cube complex the stabilisers of vertices at infinity are amenable.

Keywords: Property A; Coarse geometry; CAT(0) cube complexes; Amenability

0. Introduction

This paper is devoted to the study of Property A for finite-dimensional CAT(0) cube complexes. These spaces, which are higher-dimensional analogues of trees, appear naturally in many problems in geometric group theory and low-dimensional topology [2,7,13,19,21]. Property A was introduced by Yu as a non-equivariant generalisation of amenability from the context of groups to the context of discrete metric spaces. It was used with great effect in his attack on the

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Baum–Connes conjecture, in which he proved, among other things, that Gromov's δ-hyperbolic spaces, and hence hyperbolic groups, satisfy Property A, even though they may be very far from amenable [22].

In this paper we prove:

Theorem. *Let X be a finite-dimensional* CAT(0) *cube complex. Equipped with the geodesic metric, X has Property A. The vertex set of X, equipped with the edge-path metric has Property A.*

The proof of the theorem rests on the often used statement that intervals in a CAT(0) cube complex admit combinatorial embeddings into Euclidean spaces. While this fact appears several times in the literature no proof has been published and we take the opportunity to provide one here. Our proof of this generalises to intervals in measured wall spaces, though we omit the details here as this is not relevant to the current application.

While interval embeddings exist they are far from unique. Any given interval may admit a large number of such embeddings in spaces of varying dimensions and the embeddings may be very different from one another. For each embedding the target interval fibres over the image, and again these fiberings vary considerably. Nonetheless it is a remarkable fact that regardless of how we embed the interval into Euclidean space the norms of the functions we are computing on each fibre are independent of the embedding chosen.

Our technique may well have other applications and we present one here. A group acting properly on an Hadamard space, a building for example, fixing a point in a suitable refinement of the visual boundary is amenable [6]. In the context of CAT(0) cube complexes the natural choice for the boundary is the combinatorial boundary.

Theorem. *A countable group acting properly on a finite-dimensional* CAT(0) *cube complex and fixing a vertex at infinity is amenable.*

The advantage to working with the combinatorial boundary rather than the refined Hadamard boundary is that it is typically much smaller. One might expect the cost of this to be somewhat larger stabilisers at infinity, however our theorem shows that this is not the case. The stabilisers at infinity in both cases are virtually abelian of rank bounded by the dimension of the cube complex.

Our main theorem is known to be false for infinite-dimensional cube complexes [16], thus our result is the best possible. While it is already known for finite-dimensional CAT(0) cube complexes admitting a cocompact action by a countable discrete group [5], the approach taken there involved a deformation of the standard embedding of the cube complex in Hilbert space and rested on a functional analytic argument involving the uniform Roe algebra to conclude Property A (see [4,12]). That approach is ultimately unsuitable for non-locally finite complexes. Here, we shall remove the assumption of local finiteness by offering a direct proof of Property A in which the asymptotically invariant functions called for in Yu's non-equivariant generalisation of the Følner criterion are explicitly constructed. Furthermore we do not require the existence of a group action to make this argument work. The problem of clarifying the relationship between Property A and coarse embeddability (in Hilbert space) has attracted some attention lately, and indeed was a motivation for our study. As a consequence of the above theorem, and the coarse invariance of Property A, we obtain the following corollaries.