Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 256 (2009) 1588–1617
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

A Sobolev-like inequality for the Dirac operator

Simon Raulot [1]

Université de Neuchâtel, Institut de Mathématiques, Rue Emile-Argand 11, 2007 Neuchâtel, Switzerland

Received 26 May 2008; accepted 11 November 2008

Available online 28 November 2008

Communicated by Paul Malliavin

Abstract

In this article, we prove a Sobolev-like inequality for the Dirac operator on closed compact Riemannian spin manifolds with a nearly optimal Sobolev constant. As an application, we give a criterion for the existence of solutions to a nonlinear equation with critical Sobolev exponent involving the Dirac operator. We finally specify a case where this equation can be solved.

Keywords: Dirac operator; Sobolev inequality; Conformal geometry; Nonlinear elliptic equations

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geqslant 3$. The Sobolev embedding theorem asserts that the Sobolev space H_1^2 of functions $u \in L^2$ such that $\nabla u \in L^2$ embeds continuously in the Lebesgue space L^N (with $N = \frac{2n}{n-2}$). In other words, there exist two constants $A, B > 0$ such that, for all $u \in H_1^2$, we have

$$\left(\int_M |u|^N \, dv(g)\right)^{\frac{2}{N}} \leqslant A \int_M |\nabla u|^2 \, dv(g) + B \int_M u^2 \, dv(g). \qquad S(A, B)$$

E-mail address: simon.raulot@unine.ch.

[1] Supported by the Swiss SNF grant 20-118014/1.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.11.007

Considerable work has been devoted to the analysis of sharp Sobolev-type inequalities, very often in connection with concrete problems from geometry. One of these concerns the best constant in $S(A,B)$ defined by

$$A_2(M) := \inf \mathcal{A}_2(M),$$

where

$$\mathcal{A}_2(M) := \{A > 0 \,/\, \exists B > 0 \text{ such that } S(A,B) \text{ holds for all } u \in C^\infty(M)\}.$$

From $S(A,B)$ and by definition of A_2, we easily get that:

(1) $A_2(M) \geqslant K(n,2)^2$,
(2) for any $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that inequality $S(A_2(M)+\varepsilon, B_\varepsilon)$ holds.

Here $K(n,2)^2$ denotes the best constant of the corresponding Sobolev embedding theorem in the Euclidean space given by (see [8,33]):

$$K(n,2)^2 = \frac{4}{n(n-2)\omega_n^{2/n}},$$

where ω_n stands for the volume of the standard n-dimensional sphere. In fact, Aubin [8] showed that $A_2(M) = K(n,2)^2$ and conjectured that $S(A,B)$ should hold for $A = K(n,2)^2$, that is $\mathcal{A}_2(M)$ is closed. The proof of this conjecture by Hebey and Vaugon (see [22,23]) gave rise to various interesting problems dealing with the best constants in Riemannian Geometry. One of those given in [24], is the problem of prescribed critical functions which study the existence of functions for which $S(A_2(M), B_0)$ is an equality (here $B_0 > 0$ denotes the infimum on $B > 0$ such that $S(A_2(M), B_0)$ holds). For more details and related topics, we refer to [16].

Recall that one of the first geometric applications of the best constant problem has been discovered by Aubin [7] regarding the Yamabe problem. This famous problem of Riemannian geometry can be stated as follows: given a compact Riemannian manifold (M^n, g) of dimension $n \geqslant 3$, can one find a metric conformal to g such that its scalar curvature is constant? This problem has a long and fruitful history and it has been completely solved in several steps by Yamabe [36], Trudinger [35], Aubin [7] and finally Schoen [31] using the Positive Mass Theorem coming from General Relativity (see also [27] for a complete review). The Yamabe problem is in fact equivalent to find a smooth positive solution $u \in C^\infty(M)$ to a nonlinear elliptic equation:

$$L_g u := 4\frac{n-1}{n-2}\Delta_g u + R_g u = \lambda u^{N-1}, \tag{1}$$

where L_g is known as the conformal Laplacian (or the Yamabe operator), Δ_g (resp. R_g) denotes the standard Laplacian acting on functions (resp. the scalar curvature) with respect to the Riemannian metric g and $\lambda \in \mathbb{R}$ is a constant. Indeed, if such a function exists then the metric $\overline{g} = u^{N-2} g$ is conformal to g and satisfies $R_{\overline{g}} = \lambda$. A standard variational approach cannot allow to conclude because of the lack of compactness in the Sobolev embedding theorem involved in this method. However, Aubin [7] proved that if: