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On dispersive properties of discrete 2D Schrödinger and Klein–Gordon equations

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Abstract

We derive the long-time asymptotics for solutions of the discrete 2D Schrödinger and Klein–Gordon equations.

Keywords: Discrete Schrödinger and Klein–Gordon equations; Lattice; Cauchy problem; Long-time asymptotics

1. Introduction

In this paper we establish the long-time behavior of solutions to the discrete two-dimensional Schrödinger and Klein–Gordon equations. We extend a general strategy introduced by Vainberg [14], Jensen, Kato [6], Murata [10] concerning the wave, Klein–Gordon and Schrödinger equations, to the discrete case. Namely, we establish the smoothness of the resolvent of a stationary problem at the nonsingular points of continuous spectrum, and a generalised 'Puiseux expansion' at the singular points which are critical values of the symbol. Then, the long-time asymptotics can be obtained by means of the (inverse) Fourier–Laplace transform.

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We restrict ourselves to the "nonsingular case," in the sense of [10], where the truncated resolvent is bounded at the singular points of the continuous spectrum, i.e. there are no resonances or eigenvalues. This holds generically and allows us to get decay of order $\sim t^{-1}(\log t)^{-2}$ which is desirable for applications to scattering problems.

First, we consider discrete version of the 2D Schrödinger equation

$$\begin{cases} i\dot{\psi}(x,t) = H\psi(x,t) := \big(-\Delta + V(x)\big)\psi(x,t), \\ \psi|_{t=0} = \psi_0, \end{cases} \Bigg| \quad x \in \mathbb{Z}^2,\ t \in \mathbb{R}. \tag{1.1}$$

Here Δ stands for the difference Laplacian in $\mathbb{Z}^2$ defined by

$$\Delta\psi(x) = \sum_{|x-y|=1} \psi(y) - 4\psi(x), \quad x \in \mathbb{Z}^2, \tag{1.2}$$

for functions $\psi : \mathbb{Z}^2 \to \mathbb{C}$.

Definition 1.1. Denote by $\mathcal{V}$ the set of real-valued functions f on the lattice $\mathbb{Z}^2$ with finite support.

Assume that $V \in \mathcal{V}$. If we apply the Fourier–Laplace transform

$$\tilde{\psi}(x,\omega) = \int_0^\infty e^{i\omega t}\psi(x,t)\,dt, \quad \operatorname{Im}\omega > 0, \tag{1.3}$$

to (1.1), then we obtain the stationary equation

$$(H - \omega)\tilde{\psi}(\omega) = -i\psi_0, \quad \operatorname{Im}\omega > 0. \tag{1.4}$$

Note that the integral (1.3) converges, since $\|\psi(\cdot,t)\|_{l^2} = const$ by charge conservation. Hence

$$\tilde{\psi}(\cdot,\omega) = -iR(\omega)\psi_0, \quad \operatorname{Im}\omega > 0, \tag{1.5}$$

where $R(\omega) = (H-\omega)^{-1}$ is the resolvent of the Schrödinger operator H.

We are going to use functional spaces which are discrete versions of the Agmon spaces [1]. These spaces are the weighted Hilbert spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z}^2)$ with the norm

$$\|u\|_{l^2_\sigma} = \big\|\big(1+x^2\big)^{\sigma/2}u\big\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Let us denote by

$$B(\sigma,\sigma') = \mathcal{L}\big(l^2_\sigma, l^2_{\sigma'}\big), \qquad \mathbf{B}(\sigma,\sigma') = \mathcal{L}\big(l^2_\sigma \oplus l^2_\sigma, l^2_{\sigma'} \oplus l^2_{\sigma'}\big)$$

the spaces of bounded linear operators from l^2_σ to $l^2_{\sigma'}$ and from $l^2_\sigma \oplus l^2_\sigma$ to $l^2_{\sigma'} \oplus l^2_{\sigma'}$, respectively.

Note that the continuous spectrum of the operator H coincides with the interval $[0,8]$, and the kernel of the resolvent has singularities of the logarithmic type at points $\omega_1 = 0$, $\omega_2 = 4$, and $\omega_3 = 8$. The points are critical values of the symbol $4(\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2})$, $(\theta_1\theta_2) \in T^2$ of the