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Reduced Weyl asymptotics for pseudodifferential operators on bounded domains II. The compact group case ☆

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Abstract

Let $G \subset \mathrm{O}(n)$ be a compact group of isometries acting on n-dimensional Euclidean space $\mathbb{R}^n$, and $\mathbf{X}$ a bounded domain in $\mathbb{R}^n$ which is transformed into itself under the action of G. Consider a symmetric, classical pseudodifferential operator A_0 in $\mathrm{L}^2(\mathbb{R}^n)$ that commutes with the regular representation of G, and assume that it is elliptic on $\mathbf{X}$. We show that the spectrum of the Friedrichs extension A of the operator $\mathrm{res} \circ A_0 \circ \mathrm{ext} : \mathrm{C}_c^\infty(\mathbf{X}) \to \mathrm{L}^2(\mathbf{X})$ is discrete, and using the method of the stationary phase, we derive asymptotics for the number $N_\chi(\lambda)$ of eigenvalues of A equal or less than λ and with eigenfunctions in the χ-isotypic component of $\mathrm{L}^2(\mathbf{X})$ as $\lambda \to \infty$, giving also an estimate for the remainder term for singular group actions. Since the considered critical set is a singular variety, we recur to partial desingularization in order to apply the stationary phase theorem.

Keywords: Pseudodifferential operators; Asymptotic distribution of eigenvalues; Compact group actions; Peter–Weyl decomposition; Partial desingularization

Contents

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1. Introduction

Let $G \subset \mathrm{O}(n)$ be a compact Lie group of isometries acting on Euclidean space $\mathbb{R}^n$, and $\mathbf{X}$ a bounded open set of $\mathbb{R}^n$ which is transformed into itself under the action of G. Consider the regular representation of G

$$T(k)\varphi(x) = \varphi\big(k^{-1}x\big) \tag{1}$$

in the Hilbert spaces $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathrm{L}^2(\mathbf{X})$ of square-integrable functions by unitary operators. As a consequence of the Peter–Weyl theorem, T decomposes into isotypic components according to

$$\mathrm{L}^2\big(\mathbb{R}^n\big) = \bigoplus_{\chi \in \hat{G}} \mathcal{H}_\chi, \qquad \mathrm{L}^2(\mathbf{X}) = \bigoplus_{\chi \in \hat{G}} \operatorname{res} \mathcal{H}_\chi,$$

where $\hat{G}$ denotes the set of irreducible characters of G, and $\operatorname{res}: \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbf{X})$ is the natural restriction operator. The spaces $\mathcal{H}_\chi$ are closed subspaces, and the corresponding orthogonal projection operators are given by

$$P_\chi = d_\chi \int_G \overline{\chi(k)} T(k)\, dk, \tag{2}$$

where $d_\chi = \chi(\mathbf{1})$ is the dimension of any irreducible representation belonging to the character χ, and dk denotes the normalized Haar measure on G. In what follows, we do not assume that the boundary $\partial\mathbf{X}$ of $\mathbf{X}$ is smooth, but only that there exists a constant $c > 0$ such that for any sufficiently small $\varrho > 0$, $\operatorname{vol}(\partial\mathbf{X})_\varrho \leqslant c\varrho$, where $(\partial\mathbf{X})_\varrho = \{x \in \mathbb{R}^n:\ \operatorname{dist}(x, \partial\mathbf{X}) < \varrho\}$, and that $0 \notin \partial\mathbf{X}$.

Consider now a symmetric, classical pseudodifferential operator A_0 in $\mathbb{R}^n$ of order $2m$ that commutes with the operators $T(k)$ for all $k \in G$. Let a_{2m} be its principal symbol, and assume that there exists a constant $C_0 > 0$ such that

$$a_{2m}(x, \xi) \geqslant C_0 |\xi|^{2m}, \quad \forall x \in \mathbf{X},\ \forall \xi \in \mathbb{R}^n. \tag{3}$$

If we write $\operatorname{ext}: \mathrm{C}_\mathrm{c}^\infty(\mathbf{X}) \to \mathrm{L}^2(\mathbb{R}^n)$ for the natural extension operator by zero, it turns out that under condition (3), the operator

$$\operatorname{res} \circ A_0 \circ \operatorname{ext}: \mathrm{C}_\mathrm{c}^\infty(\mathbf{X}) \to \mathrm{L}^2(\mathbf{X})$$