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A Poincaré inequality on loop spaces

Xin Chen, Xue-Mei Li *, Bo Wu

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

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Abstract

We show that the Laplacian on the loop space over a class of Riemannian manifolds has a spectral gap. The Laplacian is defined using the Levi-Civita connection, the Brownian bridge measure and the standard Bismut tangent spaces.

Keywords: Path space; Loop space; Brownian bridge measure; Poincaré inequalities; Malliavin calculus

1. Introduction

Let M be a complete Riemannian manifold and $x_0 \in M$ fixed. For $T > 0$ let $\mathcal{C}_{x_0}M$ and $L_{x_0}M \equiv \mathcal{C}_{x_0,x_0}M$ denote the space of continuous paths or the space of continuous loops based at x_0, so

$$\mathcal{C}_{x_0}M = \{\sigma : [0, T] \to M \mid \sigma \text{ is continuous}, \ \sigma(0) = x_0\}$$

and $L_{x_0}M = \{\sigma \in \mathcal{C}_{x_0}M, \ \sigma(T) = x_0\}$. We are concerned in establishing an L^2 theory which relates to the geometry and the topology of the path space and its subspaces. For the Wiener space there is the canonically defined Cameron–Martin space of the Gaussian measure and the associated gradient operator. The corresponding gradient operator gives rise to the Ornstein–Uhlenbeck operator and a well formulated L^2 theory. The challenge for a general path space is

* Corresponding author.
E-mail address: xuemei@xuemei.org (X.-M. Li).

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the lack of a natural choice of subspaces of the tangent spaces with Hilbert space structure and a natural measure. The aim here is to analyse properties of a suitably defined Laplacian d^*d for d on an L^2 space of functions with range in the dual space of a suitably defined sub-bundle of the tangent bundle and d^* its L^2 dual. The exponential map which defines the manifold structure depends on the choice of linear connections on the underlying space. This will determine an unbounded linear operator:

$$d : L^2(\mathcal{C}_{x_0}M; \mathbf{R}) \to L^2\Gamma H^*$$

to the space of L^2 differential H one-forms. Despite that the question of the uniqueness of the operator $\mathcal{L}$ remains open we study here the spectrum properties of the operator with the initial domain the space of smooth cylindrical functions.

A Poincaré inequality on a space N, $\int_N (f-\bar{f})^2\mu(dx) \leqslant \frac{1}{C}\int_N |\nabla f|^2\mu(dx)$, depends on the gradient like operator ∇, an admissible set of real valued functions on N and a finite measure μ on N which is normalized to have total mass 1. Here $\bar{f} = \int f\,d\mu$. For a compact Riemannian manifold and the usual gradient the Poincaré constant C, the first non-trivial eigenvalue of the Laplacian, is related to the isoperimetric constant in Cheeger's isoperimetric inequality $h = \inf_A \frac{\mu(\partial A)}{\min\{\mu(A),\mu(M/A)\}}$, where the infimum is taken over all open subsets of M. Standard isoperimetric inequalities say that for an open bounded set A in $\mathbf{R}^n$, the ratio between the area of its boundary ∂A and the volume of A to the power of $1-\frac{1}{n}$ is minimized by the unit ball. In relation to Poincaré inequality, especially in infinite dimensions, the more useful form of isoperimetric inequality is that of Cheeger. It was shown by Cheeger [15] that $h^2 \leqslant 4\lambda_1$ and if K is the lower bound of the Ricci curvature Buser [12] showed that $\lambda_1 \leqslant C(\sqrt{K}h + h^2)$ for which M. Ledoux [35] has a beautiful analytic proof. For such inequalities for Gaussian measures see e.g. Ledoux [36] and Ledoux–Talagrand [37].

As Wiener measure and the Brownian bridge measure on the Wiener space are Gaussian measures, logarithmic Sobolev inequalities (L.S.I.) hold. The classical approach to this is to use the symmetric property, rotation invariance, of the Gaussian measure or the commutation relation of the Ornstein–Uhlenbeck semi-group. A number of simple proofs have since been given. It is L. Gross, [31], who obtained the logarithmic Sobolev inequality and remarked on its validity in an infinite dimensional space and its relation with Nelson's hypercontractivity. A L.S.I. was shown to hold for the Brownian motion measure on the path space over a compact manifold or for manifolds with suitable conditions on the Ricci curvature and using gradient operator defined by the Bismut tangent space. See Aida–Elworthy [7], Fang [27], and Hsu [34]. However, as noted in Gross [32] for non-Euclidean spaces there is a fundamental difference between Brownian motion measure and the Brownian bridge measure, for example Poincaré inequalities do not hold for the Brownian bridge measure on the Lie group S^1 due to the lack of connectedness of the loop space. We now consider the loop space with the standard Brownian bridge measure and the standard Bismut tangent space structure. There are few positive results, apart from one in [5] which states that L.S.I. holds on loop spaces over manifolds which are diffeomorphic to $\mathbf{R}^n$ and whose Riemannian metric is asymptotically flat and whose curvature satisfies certain additional conditions. In Aida [1] it was shown that the kernel of the OU operator, on a simply connected manifold, contains only constant functions. In [23] A. Eberle gave a criterion for the Poincaré inequality, for the gradient operator restricted to a single homotopy class of loops, to fail. Let γ be a closed geodesic belonging to the trivial homotopy class of a compact connected manifold (which in general may or may not exist) and let U be an ϵ-neighbourhood of γ, where ϵ is