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The spectral bounds of the discrete Schrödinger operator

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Abstract

Let $H(\lambda) = -\Delta + \lambda b$ be a discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ with a potential b and a non-negative coupling constant λ. When $b \equiv 0$, it is well known that $\sigma(-\Delta) = [0, 4d]$. When $b \not\equiv 0$, let $s(-\Delta + \lambda b) := \inf \sigma(-\Delta + \lambda b)$ and $M(-\Delta + \lambda b) := \sup \sigma(-\Delta + \lambda b)$ be the bounds of the spectrum of the Schrödinger operator. One of the aims of this paper is to study the influence of the potential b on the bounds 0 and $4d$ of the spectrum of $-\Delta$. More precisely, we give a necessary and sufficient condition on the potential b such that $s(-\Delta + \lambda b)$ is strictly positive for λ small enough. We obtain a similar necessary and sufficient condition on the potential b such that $M(-\Delta + \lambda b)$ is lower than $4d$ for λ small enough. In dimensions $d = 1$ and $d = 2$, the situation is more precise. The following result was proved by Killip and Simon (2003) (for $d = 1$) in [5], then by Damanik et al. (2003) (for $d = 1$ and $d = 2$) in [3]:

$$\text{If } \sigma(-\Delta + b) \subset [0, 4d], \text{ then } b \equiv 0.$$

Our study on the bounds of the spectrum of $(-\Delta + b)$ allows us to give a different and easy proof to this result.

Keywords: Schrödinger; Discrete; Spectrum; Bound

1. Introduction

In this paper, we study some spectral properties of the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$. This work is motivated by some results obtained in the continuous case. We first give

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an overview of these results. Let $-\Delta$ denote the non-negative Laplace operator on $L^2(\mathbb{R}^d)$. It is well known that $-\Delta$ is self-adjoint and $\sigma(-\Delta) = [0, +\infty)$. Let $V \in L^\infty(\mathbb{R}^d)$ be a real-valued potential. Then the Schrödinger operator $H := -\Delta + V$ is well defined and is self-adjoint on $L^2(\mathbb{R}^d)$. We note $s(-\Delta + V) := \inf \sigma(-\Delta + V)$ its spectral bound. A question of interest is for which potential V we have $s(-\Delta + V) > 0$. Indeed, the fact that $s(-\Delta + V) > 0$ assures an exponential decay in time of the solution to the heat equation

$$\begin{cases} \dfrac{\partial u(t,.)}{\partial t} = \Delta u(t,.) - V u(t,.), \quad t > 0, \\ u(0,.) = f \in L^2(\mathbb{R}^d). \end{cases}$$

For bounded non-negative potentials with compact support, it is easy to see from the variational formula

$$s(-\Delta + V) = \inf_{\substack{u \in D(-\Delta+V) \\ \|u\|=1}} \int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} V u^2$$

that $s(-\Delta + V) = 0$. To assure the strict positivity of the spectral bound of the Schrödinger operator, the potential must have a contribution in all the space in some sense. Indeed, W. Arendt and C.J.K. Batty [1] proved that $s(-\Delta + V) > 0$ holds if and only if the potential V satisfies the following mean condition $(M_{\delta,R})$:

$$\text{There exist } \delta > 0 \text{ and } R > 0 \text{ such that } \int_{B(x,R)} V \geqslant \delta \text{ for all } x \text{ in } \mathbb{R}^d. \qquad (M_{\delta,R})$$

The hypothesis of boundedness is crucial. In fact, for unbounded non-negative potentials, this characterization holds for $d = 1$ and $V \in L^1_{loc}(\mathbb{R})$ but the situation changes for higher dimensions ($d \geqslant 2$). See [1] for a counter-example in dimension $d = 2$. See also [2] for more results on the asymptotic behavior of the spectral bound of the Schrödinger operator.

Note also related results obtained by Gesztesy, Graf and Simon in [4]. Here, the authors are interested in the value of $s'(0)$ and their study also involves mean values of the potential.

For bounded potentials with positive and negative parts, $V = V^+ - V^-$, the situation is different but the same mean condition appears. Indeed, the following result was shown by E.M. Ouhabaz [9] (in a more general context of Riemannian manifolds). The spectral bound $s(-\Delta + \lambda V)$ is strictly positive for $\lambda > 0$ and small enough if and only if the positive part of the potential V^+ satisfies the mean condition $(M_{\delta,R})$ (under the condition that the negative part V^- vanishes at infinity). Note that in that paper, [9] gives conditions which characterize the class of Riemannian manifolds for which the result holds. Of course, these conditions are satisfied when the manifold is $\mathbb{R}^d$. Z. Shen [12] proved later that this result still holds for a larger class of potentials but he only studied non-negative potentials.

The aim of this paper is to study the same problem in the discrete case. Let us first recall the definition of the discrete positive Laplacian $-\Delta$ on $\ell^2(\mathbb{Z}^d)$. For all n in $\mathbb{Z}^d$, we define:

$$-\Delta(u)(n) := \sum_{\substack{m \in \mathbb{Z}^d \\ |m-n|=1}} (u_n - u_m) := 2d u_n - \sum_{\substack{m \in \mathbb{Z}^d \\ |m-n|=1}} u_m.$$