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The weighted Monge–Ampère energy of quasiplurisubharmonic functions

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Abstract

We study degenerate complex Monge–Ampère equations on a compact Kähler manifold (X, ω). We show that the complex Monge–Ampère operator $(\omega + dd^c \cdot)^n$ is well defined on the class $\mathcal{E}(X, \omega)$ of ω-plurisubharmonic functions with finite weighted Monge–Ampère energy. The class $\mathcal{E}(X, \omega)$ is the largest class of ω-psh functions on which the Monge–Ampère operator is well defined and the comparison principle is valid. It contains several functions whose gradient is not square integrable. We give a complete description of the range of the operator $(\omega + dd^c \cdot)^n$ on $\mathcal{E}(X, \omega)$, as well as on some of its subclasses. We also study uniqueness properties, extending Calabi's result to this unbounded and degenerate situation, and we give applications to complex dynamics and to the existence of singular Kähler–Einstein metrics.

Keywords: Complex Monge–Ampère operator; Kähler manifold; ω-Plurisubharmonic functions; Weighted energy

0. Introduction

Let X be a compact connected Kähler manifold of complex dimension $n \in \mathbb{N}^*$. Let ω be a Kähler form on X. Given μ a positive Radon measure on X such that $\mu(X) = \int_X \omega^n$, we study the complex Monge–Ampère equation

$$\left(\omega + dd^c \varphi\right)^n = \mu, \qquad (\mathrm{MA}_\mu)$$

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where φ, the unknown function, is such that $\omega_\varphi := \omega + dd^c\varphi$ is a positive current. Such functions are called ω-plurisubharmonic (ω-psh for short). We refer the reader to [19] for basic properties of the set $PSH(X,\omega)$ of all such functions. Here $d = \partial + \bar\partial$ and $d^c = \frac{1}{2i\pi}(\partial - \bar\partial)$.

Complex Monge–Ampère equations have been studied by several authors over the last fifty years, in connection with questions from Kähler geometry and complex dynamics (see [1,14,17, 21,22,25,26,28] for references). The first and cornerstone result is due to S.T. Yau who proved [28] that (MA_μ) admits a solution $\varphi \in PSH(X,\omega) \cap \mathcal{C}^\infty(X)$ when $\mu = f\omega^n$ is a smooth volume form.

Motivated by applications towards complex dynamics, we need here to consider measures μ which are quite singular, whence to deal with singular ω-psh functions φ. We introduce and study a class $\mathcal{E}(X,\omega)$ of ω-psh functions for which the complex Monge–Ampère operator $(\omega + dd^c\varphi)^n$ is well defined (see Definition 1.1): following E. Bedford and A. Taylor [6] we show that the operator $(\omega + dd^c\varphi)^n$ is well defined in $X \setminus (\varphi = -\infty)$ for all functions $\varphi \in PSH(X,\omega)$; the class $\mathcal{E}(X,\omega)$ is the set of functions $\varphi \in PSH(X,\omega)$ such that $(\omega + dd^c\varphi)^n$ has full mass $\int_X \omega^n$ in $X \setminus (\varphi = -\infty)$. When $n = \dim_{\mathbb{C}} X = 1$, this is precisely the subclass of functions $\varphi \in PSH(X,\omega)$ whose Laplacian does not charge polar sets. It is striking that the class $\mathcal{E}(X,\omega)$ contains many functions whose gradient is not square integrable, hence several results to follow have no local analogue (compare [8,9]).

One of our main results gives a complete characterization of the range of the complex Monge–Ampère operator on the class $\mathcal{E}(X,\omega)$.

Theorem A. *There exists $\varphi \in \mathcal{E}(X,\omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ if and only if μ does not charge pluripolar sets.*

An important tool we use is the *comparison principle* that we establish in Section 1: we show that $\mathcal{E}(X,\omega)$ is the largest class of ω-psh functions on which the complex Monge–Ampère operator $(\omega + dd^c\cdot)^n$ is well defined and the comparison principle is valid. Another crucial tool for our study is the notion of *weighted Monge–Ampère energy*, defined as

$$E_\chi(\varphi) := \int_X (-\chi)\circ\varphi\big(\omega + dd^c\varphi\big)^n,$$

where $\chi : \mathbb{R}^- \to \mathbb{R}^-$ is an increasing function such that $\chi(-\infty) = -\infty$. The properties of this energy are quite different whether the weight χ is convex ($\chi \in \mathcal{W}^-$) or concave ($\chi \in \mathcal{W}^+$). We show (Proposition 2.2) that

$$\mathcal{E}(X,\omega) = \bigcup_{\chi \in \mathcal{W}^-} \mathcal{E}_\chi(X,\omega),$$

where $\mathcal{E}_\chi(X,\omega)$ denotes the class of functions $\varphi \in \mathcal{E}(X,\omega)$ such that $\chi(\varphi - \sup_X \varphi) \in L^1((\omega + dd^c\varphi)^n)$. At the other extreme, we show (Proposition 3.1) that

$$PSH(X,\omega) \cap L^\infty(X) = \bigcap_{\chi \in \mathcal{W}^+} \mathcal{E}_\chi(X,\omega).$$