Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 254 (2008) 1487–1521
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Continuity envelopes and sharp embeddings in spaces of generalized smoothness ☆

Dorothee D. Haroske [a,*,1], Susana D. Moura [b,2]

[a] *Mathematical Institute, Friedrich-Schiller-University Jena, D-07737 Jena, Germany*
[b] *CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal*

Received 14 May 2007; accepted 18 December 2007
Available online 1 February 2008
Communicated by C. Kenig

Abstract

We study continuity envelopes in spaces of generalized smoothness $B^{\boldsymbol{\sigma},N}_{p,q}(\mathbb{R}^n)$ and $F^{\boldsymbol{\sigma},N}_{p,q}(\mathbb{R}^n)$. The results are applied in proving sharp embedding assertions in some limiting situations.

Keywords: Spaces of generalized smoothness; Limiting embeddings; Continuity envelopes

1. Introduction

This paper continues the study of sharp embeddings and continuity envelopes in spaces of generalized smoothness begun in [17] and [6]. Moreover, as we shall immediately explain, there also appear close connections to [5] and [7] dealing with *growth envelopes* in such spaces, also related to certain '*limiting*' situations. In [17] and [6] we considered spaces of type $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, with $0 < p, q \leqslant \infty$ ($p < \infty$ for F-spaces), $\frac{n}{p} < s \leqslant 1 + \frac{n}{p}$, where Ψ is a so-called admissible function, typically of log-type near 0. Our present paper covers these results essentially.

☆ The research was partially supported by the program 'DAAD-GRICES'.
* Corresponding author.
E-mail addresses: haroske@minet.uni-jena.de (D.D. Haroske), smpsd@mat.uc.pt (S.D. Moura).
URLs: http://www.minet.uni-jena.de/~haroske (D.D. Haroske), http://www.mat.uc.pt/~smpsd/ (S.D. Moura).
1 Also supported by the DFG Heisenberg fellowship HA 2794/1-1.
2 Also supported by Centro de Matemática da Universidade de Coimbra.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.12.009

The study of spaces of generalized smoothness has a long history, resulting on one hand from the interpolation side (with a function parameter), see [21] and [9], whereas the rather abstract approach (approximation by series of entire analytic functions and coverings) was independently developed by Gol'dman and Kalyabin in the late 1970s and early 1980s of the last century; we refer to the survey [19] and the appendix [20] which cover the extensive (Russian) literature at that time. We rely on the Fourier-analytical approach as presented in [13] recently. There one can also find a reason for the revived interest in the study of such spaces: its connection with applications for pseudo-differential operators (as generators of sub-Markovian semi-groups). Plainly these latter applications and also the topic in its full generality are out of the scope of the present paper. Likewise we only want to mention that the increased interest in function spaces of generalized smoothness is also connected with the study of trace spaces on fractals, that is, so-called h-sets Γ.

Roughly speaking, spaces of generalized smoothness extend 'classical' spaces of Besov or Triebel–Lizorkin type in the sense that the partition of $\mathbb{R}^n$ is not necessarily based on annuli, and, accordingly, the weight factor connected with the assumed smoothness of the distribution, can vary in a wider sense, that is, the 'classical' smoothness $s \in \mathbb{R}$ is replaced by a certain sequence $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$. In the special case of sequences $\boldsymbol{\sigma} = (2^{js})_{j\in\mathbb{N}_0}$, $s \in \mathbb{R}$ and $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$ we have the coincidence $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$, but the new setting is by no means restricted to this case. We benefit in this paper from a result in [24] saying that under certain assumptions on the involved sequences,

$$\big\| f \big| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \big\|^{(k)} := \| f | L_p \| + \left(\sum_{j=0}^{\infty} \sigma_j^q \, \omega_k\big(f, N_j^{-1}\big)_p^q \right)^{1/q}$$

(with the usual modification if $q = \infty$) is an equivalent quasi-norm in $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$, where $k \in \mathbb{N}$ must be sufficiently large.

In contrast to the notion of spaces of generalized smoothness, the study of continuity envelopes has a rather short history; this new tool was developed only recently in [15,16,27], initially intended for a more precise characterization of function spaces. It turned out, however, that it leads not only to surprisingly sharp results based on classical concepts, but allows a lot of applications, too, e.g. to the study of compact embeddings. We return to this point later. Roughly speaking, a continuity envelope $\mathfrak{E}_{\mathsf{C}}(X)$ of a function space X consists of a so-called continuity envelope function

$$\mathcal{E}^X_{\mathsf{C}}(t) \sim \sup_{\|f|X\|\leqslant 1} \frac{\omega(f,t)}{t}, \quad t > 0, \tag{1}$$

together with some '*fine index*' u_X; here $\omega(f,t)$ stands for the modulus of continuity, as usual. Forerunners of *continuity envelopes* in a wider sense are well known for decades, we refer to [27] and [16] for further details and historical comments. In the context of spaces of generalized smoothness see also [18] (in addition to the very recent contributions mentioned above).

Denoting by A either B or F, the main objective is to characterize the above continuity envelopes (1) of spaces $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ when these spaces are not continuously embedded into the Lipschitz space $\mathrm{Lip}^1(\mathbb{R}^n)$; for that reason we shall first prove a criterion for this embedding to hold.