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Improved moment estimates for invariant measures of semilinear diffusions in Hilbert spaces and applications

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Abstract

We study regularity properties for invariant measures of semilinear diffusions in a separable Hilbert space. Based on a pathwise estimate for the underlying stochastic convolution, we prove a priori estimates on such invariant measures. As an application, we combine such estimates with a new technique to prove the L^1-uniqueness of the induced Kolmogorov operator, defined on a space of cylindrical functions. Finally, examples of stochastic Burgers equations and thin-film growth models are given to illustrate our abstract result.

Keywords: Stochastic differential equations; Invariant measures; Moment estimates; Stochastic Burgers equations

1. Introduction

The aim of this work is to obtain improved moment estimates of invariant measures of semilinear stochastic evolution equations of the type

$$dX(t) = \big(AX(t) + B\big(X(t)\big)\big)\,dt + \sqrt{Q}\,dW_t, \quad t \geqslant 0, \tag{1.1}$$

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defined on a separable real Hilbert space H. Here A is a self-adjoint linear operator of negative type ω on H having a compact resolvent, B is a nonlinear function with subdomain $D(B) \subset H$. Q is a symmetric positive definite operator and $(W_t)_{t\geqslant 0}$ is a cylindrical Wiener process in H defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$.

Eq. (1.1) can be read as an abstract formulation of many partial differential equations perturbed by random noise such as stochastic reaction diffusion, Allen-Cahn, Burgers and Navier–Stokes equations. Existence and uniqueness of solutions to such equations are well studied, we refer to the monographs by Da Prato, Zabczyk [8,9], Cerrai [4] and the works [6,15]. We will be in particular interested in the situation, where (1.1) has a mild solution $X(t)$, $t \geqslant 0$, with a time-invariant distribution $\mu = \mathbb{P} \circ X(t)^{-1}$. Throughout this paper, we call such a solution a stationary mild solution and μ an invariant measure of (1.1). Given such a stationary mild solution, we will then derive in Section 3 moment estimates on its time-invariant distribution μ under appropriate assumptions on the coefficients of (1.1).

Moment estimates for invariant measures of stochastic partial differential equations have been studied quite intensively for some time. Recently, in the case where B is locally Lipschitz, the authors proved in [13] existence and moment estimates of an invariant measure μ corresponding to (1.1) under a Lyapunov type assumption on the coefficients A and B. These moment estimates have been the main tool to discuss well-posedness of the parabolic Cauchy problem corresponding to stochastic reaction diffusion or Allen-Cahn equations in $L^1(\mu)$. However, there are many important examples, e.g. the stochastic Burgers equation, that are still not covered by our analysis. The results in this paper can be seen as improved moment estimates on invariant measures to semilinear diffusions under weaker assumptions on its coefficients.

The main ingredient, to obtain our moment estimates, is a pathwise control on the stochastic convolution arising in the mild formulation of (1.1). This idea is taken from the paper [14] by Flandoli and Gatarek on stochastic Navier–Stokes equations, see also the paper [5] by Da Prato and Debussche where the same idea has been applied to the stochastic Burgers equation. We have generalized this technique and found simplified proofs to apply the same technique in an abstract context. To illustrate this result we discussed at the end examples of stochastic Burgers equations and thin-film growth models. We shall remark that the same result can be proved for stationary solutions of stochastic Navier–Stokes equations in the spirit of Flandoli and Gatarek [14].

The existence of a stationary mild solution is a rather weak assumption on Eq. (1.1) and in particular does not imply neither the existence of an associated full Markov process nor an associated transition semigroup $(P_t)_{t\geqslant 0}$. The existence of $(P_t)_{t\geqslant 0}$, however, can be obtained from the Hille–Yosida theory, in the case, where the Kolmogorov operator associated with (1.1) $(L, D(L))$ (resp. its closure on suitable test functions) generates a C_0-semigroup in $L^1(H, \mu)$. Based on the improved moment estimates on μ we will therefore study the existence (and uniqueness) of $(P_t)_{t\geqslant 0}$ in Section 4. The method which we follow here is new and different to the one presented in [19] due the fact that the drift term B is not supposed to be dissipative and the coefficients of the finite dimensional realization of L are not bounded. Hence we cannot use the classical theory by [17] to obtain uniform gradient estimates for the pseudo-resolvents associated with finite dimensional approximations of L.

Let us now specify our precise assumptions:

(H₀) A is self-adjoint, $\|e^{tA}\| \leqslant e^{-\omega t}$ for certain $\omega > 0$ and its resolvent A^{-1} (which exists) is compact.