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The nonlinear Schrödinger equation with white noise dispersion

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Abstract

Under certain scaling the nonlinear Schrödinger equation with random dispersion converges to the nonlinear Schrödinger equation with white noise dispersion. The aim of this work is to prove that this latter equation is globally well posed in L^2 or H^1. The main ingredient is the generalization of the classical Strichartz estimates. Additionally, we justify rigorously the formal limit described above.

Keywords: White noise dispersion; Strichartz estimates; Stochastic partial differential equation; Nonlinear fibre optics

1. Introduction

The following nonlinear Schrödinger equation with random dispersion describes the propagation of a signal in an optical fibre with dispersion management (see [1,2]):

$$\begin{cases} i\dfrac{dv}{dt} + \varepsilon m(t)\partial_{xx} v + \varepsilon^2 |v|^2 v = 0, & x \in \mathbb{R},\ t > 0, \\ v(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

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Recall that in the context of fibre optics, x corresponds to the retarded time while t corresponds to the distance along the fibre. The coefficient $\varepsilon m(t)$ accounts for the fact that ideally one would want a fibre with zero dispersion, in order to avoid chromatic dispersion of the signal. This is impossible to build in practice and engineers have proposed to build fibres with a small dispersion which varies along the fibre and has zero average. The case of a periodic deterministic dispersion has been studied in [21] where an averaged equation is derived. This averaged equation is then shown to possess ground states (see [21] for the case of positive residual dispersion, that is when $m(t)$ has positive average over a period, and [14] for the case of vanishing residual dispersion). Note that in this periodic setting, the nonlinear term is not of size ε^2 as such a nonlinear term would have no effect on the dynamics, the equation studied in [21] has in fact the coefficient ε in front on the nonlinearity.

In this article, we consider the case of a random dispersion, i.e. m is a centered stationary random process. As will be clear from our study, only a nonlinearity of order ε^2 is relevant in this context. In order to understand the limit as the small parameter ε goes to zero, it is natural to rescale the time variable by setting $u(t,x) = v(\frac{t}{\varepsilon}, x)$ and we obtain

$$\begin{cases} i\dfrac{du}{dt} + \dfrac{1}{\varepsilon} m\left(\dfrac{1}{\varepsilon^2}\right)\partial_{xx}u + |u|^2 u = 0, & x \in \mathbb{R},\ t > 0, \\ u(0) = u_0, & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

This model has been initially studied in [16] where a split step numerical scheme is proposed to simulate its solutions. Under classical ergodic assumptions on m, it is expected that the limiting model when ε goes to zero is the following stochastic nonlinear Schrödinger equation with white noise dispersion

$$\begin{cases} i\,du + \sigma_0 \partial_{xx} u \circ d\beta + |u|^2 u = 0, & x \in \mathbb{R},\ t > 0, \\ u(0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where β is a standard real valued Brownian motion, $\sigma_0^2 = 2\int_0^{+\infty} \mathbb{E}[m(0)m(t)]\,dt$, and $\circ$ is the Stratonovich product. In [16], the cubic nonlinearity is replaced by a nicer Lipschitz function so that the limiting equation can be easily studied using the fact that the evolution associated to the linear equation defines an isometry in all L^2 based Sobolev spaces. It is shown that the nonlinear Schrödinger equation with white noise dispersion is indeed the limit of the original problem and this result is used to prove that some numerical scheme produces good approximation result for a time step significantly higher than ε. Again, all this study is performed for an equation where a nice Lipschitz function replaces the power nonlinearity.

Our aim is to address the original equation with power nonlinearity. In fact, we study the more general equation for $\sigma > 0$:

$$\begin{cases} i\,du + \sigma_0^2 \Delta u \circ d\beta + |u|^{2\sigma} u\,dt = 0, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, & x \in \mathbb{R}^d. \end{cases}$$

Note that the sign in front of the nonlinear term $|u|^2 u$ is not important here, as it can be changed from $+1$ to -1 by changing β to $-\beta$ and u to its complex conjugate. Also, we will assume without loss of generality that $\sigma_0^2 = 1$.