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Interpolation orbits and optimal Sobolev's embeddings

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Abstract

We consider Sobolev's embeddings for spaces based on rearrangement invariant spaces (not necessarily with the Fatou property) on domains with a sufficiently smooth boundary in $\mathbb{R}^n$. We show that each optimal embedding $W^m E \subset G$, where $m < n$, can be obtained by the real interpolation of the well-known endpoint embeddings. We also give an orbital description of the optimal range space in Sobolev's embedding.

Keywords: Rearrangement invariant spaces; Interpolation spaces; Interpolation orbits; Sobolev spaces; Lorentz spaces; Embedding theorems; The Hardy type operators

0. Introduction

The construction of intermediate embeddings with the help of a pair of endpoint embeddings is one of the most natural applications of interpolation of linear operators. Indeed, if we have $X_0 \subset Y_0$ and $X_1 \subset Y_1$, then we deduce that $\mathcal{F}(X_0, X_1) \subset \mathcal{F}(Y_0, Y_1)$ for each interpolation construction (interpolation functor) $\mathcal{F}$. The situation described above exactly takes place for embeddings of full Sobolev's spaces of smooth functions to rearrangement invariant spaces. Here we

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have two sharp endpoint embeddings, $W^m\Lambda_{m/n}(\Omega) \subset L_\infty(\Omega)$ and $W^m L_1(\Omega) \subset \Lambda_{1-m/n}(\Omega)$, where the integer smoothness m is less than the dimension n of the underlying domain Ω (we suppose that Ω has a sufficiently smooth boundary). By definition, $W^m E$ consists of functions such that all partial derivatives up to the degree m belong to the rearrangement invariant space E, and Λ_α denotes the Lorentz space. It is natural that we are interested in sharp or optimal embeddings, and our estimation of an embedding from the point of view of optimality depends on the class of spaces, where we choose the spaces for comparison. Recall that the embedding $W^m E \subset G$ is said to be optimal with respect to some class of spaces if it is impossible to improve the embedding in the given class of spaces, i.e., to enlarge the space E or to decrease the space G. The most natural class of spaces in our case is the class of all rearrangement invariant spaces, however some conditions on the norms are often presumed.

From the first sight, the embeddings $\mathcal{F}(X_0, X_1) \subset \mathcal{F}(Y_0, Y_1)$ scarcely look optimal, even if the original endpoint embeddings are the best possible.

The main result of the present paper is that, if we apply a real method interpolation construction $\mathcal{F}$ to the endpoint Sobolev's embedding, we always obtain an optimal embedding with respect to the class of all interpolation rearrangement invariant spaces (the spaces which are interpolation spaces between L_1 and L_∞). Moreover, each *optimal* embedding of the form $W^m E \subset G$ is obtained by the real interpolation between the classical endpoint embeddings. The approach we used here goes up to the description of embeddings $W^m E \subset G$ in terms of the Hardy operator, considered in [13] and [6]. There is also some improvement in comparison with [13], because following [6] we deal with interpolation rearrangement invariant spaces which may have not the Fatou property that is presumed in [13]. The class of rearrangement invariant spaces without the Fatou property is very rich, e.g., see the recent paper [8], where a number of constructions of such spaces are presented. Perhaps some may regard these spaces as exotic. That is why we present simple examples of optimal embeddings for spaces which are intermediate between the classical Marcinkiewicz space M_θ and its separable part M_θ^o. Thus, we obtain some new optimal embeddings which are "invisible" from the point of view of embedding inequalities.

The essence of our approach consists of reduction to the study of linear operators, taking the couple $\{L_1, \Lambda_{m/n}\}$ to the couple $\{\Lambda_{1-m/n}, L_\infty\}$. Thus, we come to an orbital description of the optimal range spaces for Sobolev's embeddings. As an application, we consider embeddings $W^m L_{\rho,p} \subset G$, where $L_{\rho,p}$ are the Lorentz spaces with arbitrary quasi-concave function parameter ρ, such that $L_{\rho,p}$ is an intermediate space between L_1 and $\Lambda_{m/n}$. We give a simple explicit description of the optimal range spaces, similar to the limiting optimal embedding obtained in [6].

The paper is organized as follows. In Section 1, we recall rearrangement invariant spaces, orbital equivalence of intermediate spaces, and orbital equivalence of Banach couples. The embeddings of Sobolev's spaces are considered in Section 2. The formulation of our main results is given here. We present optimal embeddings, implied by the orbital description of optimal range spaces, as well as the optimal embeddings for some non-separable spaces intermediate between the Marcinkiewicz space and its separable part. Interpolation orbits are used in Section 3 for the analysis of the Hardy operator. The final Section 4 contains the proof of the main theorem.

1. Rearrangement invariant spaces. Orbital equivalence

We shall consider spaces of functions defined on bounded domains Ω in $\mathbb{R}^n$ with the Lebesgue measure. Assume $\mathrm{mes}(\Omega) = 1$.