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ScienceDirect
Journal of Functional Analysis 253 (2007) 241–272
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Boundary singularities for weak solutions of semilinear elliptic problems

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Received 5 February 2007; accepted 2 May 2007

Available online 24 July 2007

Communicated by J. Coron

Abstract

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with smooth boundary $\partial\Omega$. We construct positive weak solutions of the problem $\Delta u + u^p = 0$ in Ω, which vanish in a suitable trace sense on $\partial\Omega$, but which are singular at prescribed isolated points if p is equal or slightly above $\frac{N+1}{N-1}$. Similar constructions are carried out for solutions which are singular at any given embedded submanifold of $\partial\Omega$ of dimension $k \in [0, N-2]$, if p equals or it is slightly above $\frac{N-k+1}{N-k-1}$, and even on countable families of these objects, dense on a given closed set. The role of the exponent $\frac{N+1}{N-1}$ (first discovered by Brezis and Turner [H. Brezis, R. Turner, On a class of superlinear elliptic problems, Comm. Partial Differential Equations 2 (1977) 601–614]) for boundary regularity, parallels that of $\frac{N}{N-2}$ for interior singularities.

Keywords: Prescribed boundary singularities; Very weak solution; Critical exponents

1. Introduction and statement of main results

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$ with smooth boundary $\partial\Omega$. A model of nonlinear elliptic boundary value problem is the classical Lane–Emden–Fowler equation,

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.05.023

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $p > 1$. We are interested in finding solutions to this problem which are smooth in Ω and equal to 0 almost everywhere on $\partial\Omega$ with respect to the $(N-1)$-dimensional measure. More precisely, we want to study solutions to problem (1.1) which satisfy the boundary condition in a suitable trace sense, while not necessarily in a continuous fashion.

Following Brezis and Turner [3] and Quittner and Souplet [9], we will say that a positive function $u \in \mathcal{C}^\infty(\Omega)$ is a *very weak solution* of problem (1.1) if

$$u \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) u^p \in L^1(\Omega)$$

and if

$$\int_\Omega \big(u \Delta v + u^p v\big)\, dx = 0 \quad \text{for all } v \in \mathcal{C}^2(\bar{\Omega}) \text{ with } v = 0 \text{ on } \partial\Omega.$$

From the results in [3,9], it follows that if p satisfies the constraint

$$1 < p < \frac{N+1}{N-1} \tag{1.2}$$

then a very weak solution u is actually in $H_0^1(\Omega)$, and it is a weak solution in the usual variational sense:

$$u \in H_0^1(\Omega) \quad \text{and} \quad \int_\Omega \big(\nabla u \nabla v - u^p v\big)\, dx = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

Elliptic regularity then yields $u \in \mathcal{C}^2(\bar{\Omega})$, so that u solves (1.1) in the classical sense. As it is well known, a constrained minimization procedure involving Sobolev's embedding implies the existence of a weak-variational solution to (1.1) for $1 < p < \frac{N+2}{N-2}$. A natural question is then whether very weak solutions of (1.1) are classical within a broader range of exponents than (1.2). Partially answering this question negatively, Souplet [10] constructed an example of a positive function $a \in L^\infty(\Omega)$ such that problem (1.1), with u^p replaced by $a(x)u^p$ for $p > \frac{N+1}{N-1}$, has a very weak solution which is *unbounded*, developing a point singularity on the boundary. Thus, as far as boundary regularity of very weak solutions is concerned, the exponent $p = \frac{N+1}{N-1}$ is critical. In the same spirit, we would also like to mention the recent paper by McKenna and Reichel [7] where very weak solutions on Lipschitz domain are considered and where critical exponents depending on the local behavior of the boundary are defined, see also Beresticky, Capuzzo-Dolcetta and Nirenberg [1].

The aim of this paper is to construct solutions to problem (1.1) with prescribed singularities on the boundary. To state an important special case of our main results we need a definition:

Definition 1.1. Let $u(x)$ be a function defined in Ω and $y \in \partial\Omega$. We say that

$$u(x) \to \ell \quad \text{as } x \to y \text{ nontangentially}$$