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Qualitative properties of stationary measures for three-dimensional Navier–Stokes equations

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Abstract

The paper is devoted to studying the distribution of stationary solutions for 3D Navier–Stokes equations perturbed by a random force. Under a non-degeneracy assumption, we show that the support of such a distribution coincides with the entire phase space, and its finite-dimensional projections are minorised by a measure possessing an almost surely positive smooth density with respect to the Lebesgue measure. Similar assertions are true for weak solutions of the Cauchy problem with a regular initial function. The results of this paper were announced in the short note [A. Shirikyan, Controllability of three-dimensional Navier–Stokes equations and applications, in: Sémin. Équ. Dériv. Partielles, 2005–2006, École Polytech., Palaiseau, 2006].

Keywords: Stationary solutions; Irreducibility; Controllability; 3D Navier–Stokes system

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0. Introduction

Let us consider the 3D Navier–Stokes (NS) system

$$\partial_t u + \langle u, \nabla\rangle u - \nu\Delta u + \nabla p = f(t,x), \qquad \operatorname{div} u = 0, \quad x \in \mathbb{T}^3, \tag{0.1}$$

where $\mathbb{T}^3$ denotes the 3D torus, $u = (u_1, u_2, u_3)$ and p are unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, and f is an external force. In what follows, we assume that f is the time derivative of a random process with independent increments and sufficiently non-degenerate distribution in the space variables. Our aim is to study qualitative properties of the law of stationary weak solutions for (0.1). This question has significant importance in applications for at least two reasons. First, it is widely believed that stationary solutions corresponding to small values of viscosity can be used to describe turbulent behaviour of solutions. And, second, under some additional assumptions, a large class of weak solutions for (0.1) converge to a stationary solution as time goes to infinity. Before turning to a description of the contents of this paper, let us recall some earlier results on 3D stochastic NS equations.

Existence of weak solutions for the Cauchy problem and of stationary solutions, as well as some a priori estimates for them, was established by Bensoussan, Temam [2], Vishik, Komech, Fursikov [25], Capiński, Gątarek [6], Flandoli, Gątarek [9] and others. A first result showing the mixing character of 3D NS dynamics under non-degenerate random forcing was obtained by Flandoli [8]. He proved that if the noise is effective in all Fourier modes, then the support of any "admissible" weak solution coincides with the entire phase space. In the case of a rough white noise, Da Prato and Debussche [7] constructed a Markov semigroup concentrated on weak solutions of the 3D NS equations and established a mixing property for it. Under similar conditions, Odasso [14] proved that any solution obtained as a limit of Galerkin approximations converges exponentially to a stationary solution. Flandoli and Romito [10] have constructed a Markov selection of weak solutions and proved the irreducibility and strong Feller property for it, provided that the random perturbation is sufficiently rough. The results of this paper show that, in the case of periodic boundary conditions, non-degeneracy of the noise with respect to the first few Fourier modes ensures mixing character of the dynamics.

We now describe in more details the main result of this paper. Let us assume that the right-hand side of (0.1) has the form

$$f(t,x) = h(x) + \sum_{j=1}^{\infty} b_j \dot\beta_j(t) e_j(x), \tag{0.2}$$