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Journal of Functional Analysis 249 (2007) 477–504
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Maximal function and multiplier theorem for weighted space on the unit sphere ☆

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Received 3 December 2006; accepted 26 March 2007

Available online 4 May 2007

Communicated by G. Pisier

Abstract

For a family of weight functions invariant under a finite reflection group, the boundedness of a maximal function on the unit sphere is established and used to prove a multiplier theorem for the orthogonal expansions with respect to the weight function on the unit sphere. Similar results are also established for the weighted space on the unit ball and on the standard simplex.

Keywords: Maximal function; Multiplier; h-Harmonics; Sphere; Orthogonal polynomials; Ball; Simplex

1. Introduction

The purpose of this paper is to study the maximal function in the weighted spaces on the unit sphere and the related domains. Let $S^d = \{x\colon \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^{d+1}$, where $\|x\|$ denotes the usual Euclidean norm. Let $\langle x, y\rangle$ denote the usual Euclidean inner product. We

☆ The first author was partially supported by the NSERC Canada under grant G121211001. The second author was partially supported by the National Science Foundation under Grant DMS-0604056.
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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.03.023

consider the weighted space on S^d with respect to the measure $h_\kappa^2\,d\omega$, where $d\omega$ is the surface (Lebesgue) measure on S^d and the weight function h_κ is defined by

$$h_\kappa(x) = \prod_{v\in R_+} \bigl|\langle x, v\rangle\bigr|^{\kappa_v}, \quad x \in \mathbb{R}^{d+1}, \tag{1.1}$$

in which R_+ is a fixed positive root system of $\mathbb{R}^{d+1}$, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$, and κ is a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u, the reflection with respect to the hyperplane perpendicular to u, is conjugate to σ_v in the reflection group G generated by the reflections $\{\sigma_v\colon\ v \in R_+\}$. The function h_κ is invariant under the reflection group G. The simplest example is given by the case $G = \mathbb{Z}_2^{d+1}$ for which h_κ is just the product weight function

$$h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \quad \kappa_i \geqslant 0,\ x = (x_1, \ldots, x_{d+1}). \tag{1.2}$$

Denote by a_κ the normalization constant, $a_\kappa^{-1} = \int_{S^d} h_\kappa^2(y)\,d\omega(y)$. We consider the weighted space $L^p(h_\kappa^2; S^d)$ of functions on S^d with the finite norm

$$\|f\|_{\kappa,p} := \biggl(a_\kappa \int_{S^d} \bigl|f(y)\bigr|^p h_\kappa^2(y)\,d\omega(y)\biggr)^{1/p}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(S^d)$, the space of continuous functions on S^d with the usual uniform norm $\|f\|_\infty$.

The weight function (1.1) was first studied by Dunkl in the context of h-harmonics, which are orthogonal polynomials with respect to h_κ^2. A homogeneous polynomial is called an h-spherical harmonics if it is orthogonal to all polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2; S^d)$. The theory of h-harmonics is in many ways parallel to that of ordinary harmonics (see [5]). In particular, many results on the spherical harmonics expansions have been extended to h-harmonics expansions, see [3–5,8,12,13] and the references therein. Much of the analysis of h-harmonics depends on the intertwining operator V_κ that intertwines between Dunkl operators, which are a commuting family of first order differential–difference operators, and the usual partial derivatives. The operator V_κ is a uniquely determined positive linear operator. To see the importance of this operator, let $\mathcal{H}_n^{d+1}(h_\kappa^2)$ denote the space of h-harmonics of degree n; the reproducing kernel of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ can be written in terms of V_κ as

$$P_n^h(x, y) = \frac{n + \lambda_k}{\lambda_\kappa} V_\kappa\bigl[C_n^{\lambda_k}\bigl(\langle x, \cdot\rangle\bigr)\bigr](y), \quad x, y \in S^d, \tag{1.3}$$

where C_n^λ is the nth Gegenbauer polynomial, which is orthogonal with respect to the weight function $w_\lambda(t) := (1 - t^2)^{\lambda - 1/2}$ on $[-1, 1]$, and

$$\lambda_\kappa = \gamma_\kappa + \frac{d-1}{2} \quad \text{with } \gamma_\kappa = \sum_{v\in R_+} \kappa_v. \tag{1.4}$$