Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 257 (2009) 3802–3822
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Ground state solutions for some indefinite variational problems

Andrzej Szulkin [a,1], Tobias Weth [b,*]

[a] *Department of Mathematics, Stockholm University, 106 91 Stockholm, Sweden*
[b] *Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, D-60054 Frankfurt, Germany*

Received 24 October 2008; accepted 16 September 2009

Available online 3 October 2009

Communicated by H. Brezis

Abstract

We consider the nonlinear stationary Schrödinger equation $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$. Here f is a superlinear, subcritical nonlinearity, and we mainly study the case where both V and f are periodic in x and 0 belongs to a spectral gap of $-\Delta + V$. Inspired by previous work of Li et al. (2006) [11] and Pankov (2005) [13], we develop an approach to find ground state solutions, i.e., nontrivial solutions with least possible energy. The approach is based on a direct and simple reduction of the indefinite variational problem to a definite one and gives rise to a new minimax characterization of the corresponding critical value. Our method works for merely continuous nonlinearities f which are allowed to have weaker asymptotic growth than usually assumed. For odd f, we obtain infinitely many geometrically distinct solutions. The approach also yields new existence and multiplicity results for the Dirichlet problem for the same type of equations in a bounded domain.

Keywords: Schrödinger equation; Strongly indefinite functional; Minimax principle; Ground state

* Corresponding author.
E-mail addresses: andrzejs@math.su.se (A. Szulkin), weth@math.uni-frankfurt.de (T. Weth).
1 Supported in part by the Swedish Research Council.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.09.013

1. Introduction

In this paper we will be concerned with the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.1}$$

Here $H^1(\mathbb{R}^N)$ is the usual Sobolev space. Our assumptions on V and f stated below imply that the Schrödinger operator $-\Delta + V$ is selfadjoint and semibounded in $L^2(\mathbb{R}^N)$ and solutions of (1.1) are critical points of the functional

$$\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R}), \quad \Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx, \tag{1.2}$$

where F is the primitive of f with respect to u. In the last part of the paper, we will also consider a related variational problem associated with a semilinear elliptic boundary value problem in a bounded domain. We will be mainly interested in the case where these problems are indefinite in the sense that 0 is not a local minimum for the corresponding functionals, but some of our results are new also in the definite case. In the case of the full space problem (1.1) we focus on periodic data – another setting will be discussed briefly in Section 4 below. For the Schrödinger operator $-\Delta + V$ we assume:

(S_1) V is continuous, 1-periodic in $x_1, \dots, x_N$ and $0 \notin \sigma(-\Delta + V)$, the spectrum of $-\Delta + V$.

Starting with the seminal work of Angenent [2], Coti Zelati and Rabinowitz [6], and Alama and Li [1], this case has attracted immense attention in the last 15 years. Setting $F(x,u) := \int_0^u f(x,s)\,ds$, we suppose that f satisfies the following assumptions:

(S_2) f is continuous, 1-periodic in $x_1, \dots, x_N$ and $|f(x,u)| \leqslant a(1 + |u|^{p-1})$ for some $a > 0$ and $p \in (2, 2^*)$, where $2^* := 2N/(N-2)$ if $N \geqslant 3$ and $2^* := +\infty$ if $N = 1$ or 2.
(S_3) $f(x,u) = o(u)$ uniformly in x as $|u| \to 0$.
(S_4) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$.
(S_5) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

While (S_1)–(S_3) are standard assumptions in this context, the following Ambrosetti–Rabinowitz type superlinearity condition is commonly used in place of (S_4) and (S_5):

(AR) $0 < \eta F(x,u) \leqslant f(x,u)u$ for some $\eta > 2$ and all $u \in \mathbb{R} \setminus \{0\}$, $x \in \mathbb{R}^N$.

We recall that (AR) implies $F(x,u) \geqslant c|u|^\eta > 0$ for $|u| \geqslant 1$ and all $x \in \mathbb{R}^N$, so it is a stronger condition than (S_4). To state our results, we fix some notation. Let $E := H^1(\mathbb{R}^N)$. By (S_1) there is an equivalent inner product $\langle \cdot, \cdot \rangle$ in E such that

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx,$$