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Sufficient conditions for the projective freeness of Banach algebras

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Abstract

Let R be a unital semi-simple commutative complex Banach algebra, and let $M(R)$ denote its maximal ideal space, equipped with the Gelfand topology. Sufficient topological conditions are given on $M(R)$ for R to be a projective free ring, that is, a ring in which every finitely generated projective R-module is free. Several examples are included, notably the Hardy algebra $H^\infty(X)$ of bounded holomorphic functions on a Riemann surface of finite type, and also some algebras of stable transfer functions arising in control theory.

Keywords: Projective free ring; Banach algebra; Maximal ideal space

1. Introduction

The aim of this article is to give sufficient conditions on the maximal ideal space of a Banach algebra R which guarantee that R is a projective free ring. (Throughout the paper, we will consider only complex semi-simple commutative unital Banach algebras.) The precise definition of a projective free ring is recalled below.

Definition 1.1. Let R be a commutative ring with identity. The ring R is said to be *projective free* if every finitely generated projective R-module is free. Recall that if M is an R-module, then

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(1) M is called *free* if $M \cong R^d$ for some integer $d \geqslant 0$;
(2) M is called *projective* if there exists an R-module N and an integer $d \geqslant 0$ such that $M \oplus N \cong R^d$.

In terms of matrices (see [4, Proposition 2.6]), the ring R is projective free iff every square idempotent matrix F is conjugate (by an invertible matrix) to a matrix of the form

$$\mathrm{diag}(I_k, 0) := \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.$$

From the matricial definition, we see for example that any field $\mathbf{k}$ is projective free, since matrices F satisfying $F^2 = F$ are diagonalizable over $\mathbf{k}$. Quillen and Suslin (see [12]) proved, independently, that the polynomial ring over a projective free ring is again projective free. From this we obtain that the polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$ is projective free. Also, if R is any projective free ring, then the formal power series ring $R[\![x]\!]$ in a central indeterminate x is again projective free [5, Theorem 7]. So it follows that the ring of formal power series $\mathbf{k}[\![x_1, \ldots, x_n]\!]$ is also projective free.

However, very little seems to be known about the projective freeness of topological rings arising in analysis. H. Grauert [8] proved that the ring $H(\mathbb{D}^n)$ of holomorphic functions on the polydisk

$$\mathbb{D}^n := \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n \;\big|\; |z_k| < 1,\ k = 1, \ldots, n\big\}$$

is projective free. In fact, from Grauert's celebrated theorem [9], by the method of the proof of our Theorem 1.2, one obtains that the ring $H(X)$ of holomorphic functions on a connected reduced Stein space X satisfying the property that any holomorphic vector bundle of finite rank on X is topologically trivial, is projective free. For instance, this is the case if the space X is contractible or if it is biholomorphic to a connected noncompact (possibly singular) Riemann surface.

In this article, we formulate certain topological conditions on the maximal ideal space of a Banach algebra R that guarantee the projective freeness of R. Our motivation arises from a result of Lin [13, Theorem 3] (see also Tolokonnikov [21, Theorem 4]), which says that the contractibility of the maximal ideal space $M(R)$ implies that the Banach algebra R is a Hermite ring. The concept of a Hermite ring is a weaker notion than that of a projective free ring. Indeed, a commutative ring R with identity is said to be *Hermite* if every finitely generated stably free R-module is free. Recall that a R-module M is called *stably free* if there exist nonnegative integers n, d such that $M \oplus R^n = R^d$. Clearly every stably free module is projective, and so every projective free ring is Hermite. In this article we prove, in particular, that contractibility of the maximal ideal space of a Banach algebra in fact implies not just Hermiteness, but also projective freeness.

In the subsequent results the maximal ideal space $M(R)$ of a semi-simple commutative unital complex Banach algebra R is considered with the Gelfand topology. Our main results are the following:

Theorem 1.2. *Let R be a semi-simple commutative unital complex Banach algebra. If the Banach algebra $C(M(R))$ of complex continuous functions on $M(R)$ is a projective free ring, then R is a projective free ring.*