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Limit-periodic Schrödinger operators in the regime of positive Lyapunov exponents ☆

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Abstract

We investigate the spectral properties of discrete one-dimensional Schrödinger operators whose potentials are generated by continuous sampling along the orbits of a minimal translation of a Cantor group. We show that for given Cantor group and minimal translation, there is a dense set of continuous sampling functions such that the spectrum of the associated operators has zero Hausdorff dimension and all spectral measures are purely singular continuous. The associated Lyapunov exponent is a continuous strictly positive function of the energy. It is possible to include a coupling constant in the model and these results then hold for every non-zero value of the coupling constant.

Keywords: Limit-periodic Schrödinger operators; Singular continuous spectrum; Lyapunov exponent

1. Introduction

This paper is a part of a sequence of papers devoted to the study of spectral properties of discrete one-dimensional limit-periodic Schrödinger operators. The first paper in this sequence [7] contains results in the regime of zero Lyapunov exponents, while the present paper investigates the regime of positive Lyapunnov exponents. Our general aim is to exhibit as rich a spectral picture as possible within this class of operators. In particular, we want to show that all basic

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spectral types are possible and, in addition, in the case of singular continuous spectrum and pure point spectrum, we are interested in examples with positive Lyapunov exponents and examples with zero Lyapunov exponents. From this point of view, the present paper will, to the best of our knowledge for the first time, exhibit limit-periodic Schrödinger operators with purely singular continuous spectrum and positive Lyapunov exponents (whereas [7] had the first examples of limit-periodic Schrödinger operators with purely singular continuous spectrum and zero Lyapunov exponents). Examples with purely absolutely continuous spectrum have been known for a long time, dating back to works of Avron and Simon [2], Chulaevsky [4], and Pastur and Tkachenko [15,16] in the 1980s. These examples (must) have zero Lyapunov exponents. Examples with pure point spectrum (and positive Lyapunov exponents at least at many energies in the spectrum) can be found in Pöschel's paper [17]; compare also the work of Molchanov and Chulaevsky [13] (who have examples with zero Lyapunov exponents). In the third paper of this sequence we use Pöschel's general theorem from [17] to construct limit-periodic examples with uniform pure point spectrum within our framework (actually these examples have uniform localization of eigenfunctions); see [8].

Our study is motivated by the recent paper [1], in which Avila disproves a conjecture raised by Simon; see [19, Conjecture 8.7]. That is, he has shown that it is possible to have ergodic potentials with uniformly positive Lyapunov exponents and zero-measure spectrum. The examples constructed by Avila are limit-periodic. In fact, the paper [1] proposes a novel way of looking at limit-periodic potentials. In hindsight, this way is quite natural and provides one with powerful technical tools. Consequently, we feel that a general study of limit-periodic Schrödinger operators may be based on this new approach and we have implemented this in [7,8] and the present paper. We anticipate that further results may be obtained along these lines.

It has been understood since the early papers on limit-periodic Schrödinger operators, and more generally almost periodic Schrödinger operators, that these operators belong naturally to the class of ergodic Schrödinger operators, where the potentials are obtained dynamically, that is, by iterating an ergodic map and sampling along the iterates with a real-valued function; see [3,5, 14] for general background. Indeed, taking the closure in ℓ^∞ of the set of translates of an almost periodic function on $\mathbb{Z}$ (i.e., the *hull* of the function), one obtains a compact Abelian group with a unique translation invariant probability measure (Haar measure). In particular, the shift on the hull is ergodic with respect to Haar measure and each element of the hull may be obtained by continuous sampling (using the evaluation at the origin, for example).

As pointed out by Avila, it is quite natural to take this one step further. That is, once a compact Abelian group and a minimal translation have been fixed, one is certainly not bound to sample along the orbits merely with functions that evaluate a sequence at one point. Rather, every continuous real-valued function on the group is a reasonable sampling function. While this is quite standard in the quasi-periodic case, we are not aware of any systematic use of it in the context of limit-periodic potentials before Avila's work [1].

The ability to fix the base dynamics and independently vary the sampling functions is very useful in constructing examples of potentials and operators that exhibit a certain desired spectral feature. This has been nicely demonstrated in [1] and is also the guiding principle in our present work. As mentioned above, our main motivation is to find examples of limit-periodic Schrödinger operators with prescribed spectral type. From this point of view, the singular continuity result we prove here is the main result of the paper. However, there was additional motivation to improve the zero measure result of Avila to a zero Hausdorff dimension result. Recent work of Damanik and Gorodetski [9,10] focused on the weakly coupled Fibonacci Hamiltonian. This is an ergodic model that is not (uniformly) almost periodic. Among the results obtained in [9,10], there is a