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# Continuity modulus of stochastic homeomorphism flows for SDEs with non-Lipschitz coefficients ☆

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**Abstract**

Let $X_t(x)$ solve the following Itô-type SDE (denoted by EQ.$(\sigma, b, x)$) in $\mathbb{R}^d$

$$\mathrm{d}X_t = \sigma(X_t) \cdot \mathrm{d}W_t + b(X_t)\,\mathrm{d}t, \qquad X_0 = x \in \mathbb{R}^d.$$

Assume that for any $N > 0$ and some $C_N > 0$

$$\left|b(x) - b(y)\right| + \left\|\nabla\sigma(x) - \nabla\sigma(y)\right\| \leqslant C_N|x-y|\left(\log|x-y|^{-1} \vee 1\right), \quad |x|, |y| \leqslant N,$$

where $\nabla$ denotes the gradient, and the explosion times of EQ.$(\sigma, b, x)$ and EQ.$(\sigma, \mathrm{tr}(\nabla\sigma \cdot \sigma) - b, x)$ are infinite for each $x \in \mathbb{R}^d$. Then we prove that for fixed $t > 0$, $x \mapsto X_t^{-1}(x)$ is $\alpha(t)$-order locally Hölder continuous a.s., where $\alpha(t) \in (0, 1)$ is exponentially decreasing to zero as the time goes to infinity. Moreover, for almost all $\omega$, the inverse flow $(t, x) \mapsto X_t^{-1}(x, \omega)$ is bicontinuous.


*Keywords:* Continuity modulus; Homeomorphism flow; SDE; Non-Lipschitz

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## 1. Introduction

Consider the Itô-type stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t = \sigma(X_t) \cdot \mathrm{d}W_t + b(X_t)\,\mathrm{d}t, \\ X_0 = x \in \mathbb{R}^d, \end{cases} \tag{1}$$

driven by infinitely many Brownian motions $W := (W^1, W^2, \ldots)$ on some probability space $(\Omega, \mathcal{F}, P)$, where $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d \times l^2$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ are measurable functions. Here $l^2$ is the usual Hilbert space of real number sequences, of which the norm is denoted by $\|\cdot\|_{l^2}$. In the sequel, the norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. This equation will be simply written as EQ.$(\sigma, b, x)$.

Assume that the coefficients satisfy the following assumptions:

(A1) For some $C_0 > 0$ and any $x, y \in \mathbb{R}^d$,

$$\sum_{i=1}^{d} \left\|\sigma^i(x) - \sigma^i(y)\right\|_{l^2}^2 \leqslant C_0 |x-y|^2 \left(\log |x-y|^{-1} \vee 1\right).$$

(A2) For some $C_1 > 0$ and any $x, y \in \mathbb{R}^d$,

$$\left|b(x) - b(y)\right| \leqslant C_1 |x-y| \left(\log |x-y|^{-1} \vee 1\right).$$

Then, there is a unique strong solution $X_t(x)$ to Eq. (1) (see [15,5]). Recently, the second named author [16,17] proved that the mappings $\mathbb{R}^d \ni x \mapsto X_t(x) \in \mathbb{R}^d$ are homeomorphisms for all $t > 0$, a.s. under (A1) and (A2). In the case of $d = 1$, this result has been proved under a slightly stronger assumption by the authors in [10], but a more precise continuity modulus for the mappings $x \mapsto X_t(x)$ was given therein. In these works, the main feature is the rapid decreasing of continuity modulus of $X_t(x)$ in $x$ as $t \to +\infty$.

Let $\mathcal{H}(\mathbb{R}^d)$ be the group of homeomorphisms on $\mathbb{R}^d$, which is endowed with the topology of compact uniform convergence of both the mapping and its inverse. Let $\mathcal{H}_\alpha(\mathbb{R}^d)$ be the Hölder subspace of $\mathcal{H}(\mathbb{R}^d)$ consisting of the elements $g$ satisfying:

$$\sup_{x \neq y \in B_N} \frac{|g(x) - g(y)|}{|x-y|^\alpha} + \sup_{x \neq y \in B_N} \frac{|g^{-1}(x) - g^{-1}(y)|}{|x-y|^\alpha} < +\infty \quad \text{for all } N > 0,$$

where $\alpha \in (0,1)$ and $B_N := \{x \in \mathbb{R}^d,\ |x| \leqslant N\}$. We remark that $\mathcal{H}_\alpha(\mathbb{R}^d)$ is not a group.

One might ask:

**Question.** *Under how less conditions on $\sigma$ and $b$, can one guarantee that for fixed $t > 0$, the mapping $x \mapsto X_t(x)$ admits a Hölder continuous version in $\mathcal{H}_{\alpha(t)}(\mathbb{R}^d)$ for some $\alpha(t) \in (0,1)$ a.s.?*

The interest for studying this problem comes from the investigation of Brownian motions on the diffeomorphism groups of the unit circle and the unit disc (cf. [1,2,11,12]). For $t > 0$, let $X_t^{-1}(x)$ denote the inverse flow of $x \mapsto X_t(x)$. Since we have found the continuity modulus of $x \mapsto X_t(x)$ in our previous works (see [16, Theorem 4.1]), we only need to study the continuity