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Self-improving behaviour of inner functions as multipliers ☆

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Abstract

Let X and Y be two spaces of analytic functions in the disk, with $X \subset Y$. For an inner function θ, it is sometimes true that whenever $f \in X$ and $f\theta \in Y$, the latter product must actually be in X. We discuss this phenomenon for various pairs of (analytic) smoothness classes X and Y.

Keywords: Inner functions; Lipschitz spaces; BMO; Bloch space

1. Introduction and results

A bounded analytic function θ on the disk $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$ is said to be *inner* if $\lim_{r\to 1^-} |\theta(r\zeta)| = 1$ for m-almost all $\zeta \in \mathbb{T}$; here $\mathbb{T} := \partial\mathbb{D}$ is the unit circle and m is the normalized arclength measure on $\mathbb{T}$. Given an inner function θ, we write $\sigma(\theta)$ for its singular set on $\mathbb{T}$, also known as the *boundary spectrum* of θ; thus $\sigma(\theta)$ is the smallest closed set $E \subset \mathbb{T}$ for which θ is analytic across $\mathbb{T} \setminus E$. Equivalently, $\sigma(\theta)$ is formed by the accumulation points of the zeros of θ and the (closed) support of the associated singular measure; see [17, Chapter II].

Because a nontrivial inner function θ is extremely oscillatory near $\sigma(\theta)$, the same should be expected of (and is "usually" true for) the product $f\theta$, where f is a generic analytic function on

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$\mathbb{D}$ that is smooth, in a sense, up to $\mathbb{T}$. In some special cases, however, the product $f\theta$ happens to preserve the nice properties of f. At the same time, it is often true that *multiplication by* θ *either destroys smoothness quite drastically or does not affect it at all*. We may speak then of a certain automatic smoothness enhancement, in the sense that stronger smoothness properties of $f\theta$ can be derived from weaker ones, and this phenomenon will be our main theme.

To be more precise, suppose X and Y are two classes of analytic functions on $\mathbb{D}$, with $X \subset Y$, and let θ be an inner function. We say that θ is (X, Y)-*improving* if every function $f \in X$ satisfying $f\theta \in Y$ must actually satisfy $f\theta \in X$. Thus, saying that θ is (X, Y)-improving amounts to the implication

$$(f \in X,\ f\theta \in Y) \quad \Rightarrow \quad f\theta \in X. \tag{1.1}$$

This implication, when true, reflects the self-improving behavior of the multiplication operator $f \mapsto f\theta$—or rather of the set $\{f\theta\colon\ f \in X\} \cap Y$—referred to in the title. In what follows, we shall focus on various specific pairs (X, Y) of "smooth analytic spaces" and describe the θ's that enjoy the self-improving property (1.1).

Meanwhile, let us pause to make some immediate observations. First of all, given X and Y as above, every inner multiplier of X (i.e., every inner function θ satisfying $\theta X \subset X$) is (X, Y)-improving. In particular, finite Blaschke products are always (X, Y)-improving for a reasonable choice of X and Y. And, of course, all inner functions are (H^p, Y)-improving, where H^p is the classical Hardy space and Y is any space containing it. Secondly, if $X \subset Y_1 \subset Y_2$, then every (X, Y_2)-improving inner function is (X, Y_1)-improving. Thirdly, we mention a generalization of the preceding property: if X, Y and Z are three analytic function spaces with $X \subset Y$, then

$$\text{every } (X, Y)\text{-improving inner function is } (X \cap Z, Y \cap Z)\text{-improving.} \tag{1.2}$$

As regards our specific pairs (X, Y), one source of these is the following string of inclusions:

$$A^\alpha \subset \mathcal{A} \subset \mathrm{VMOA} \subset \mathrm{BMOA} \subset \mathcal{B}. \tag{1.3}$$

Here, $\mathcal{A}$ stands for the *disk algebra*, i.e., the set of analytic functions on $\mathbb{D}$ that admit a continuous extension to $\mathbb{D} \cup \mathbb{T}$. By A^α, with $0 < \alpha \leqslant 1$, we denote the *Lipschitz space* consisting of those $f \in \mathcal{A}$ which satisfy

$$\big|f(z_1) - f(z_2)\big| \leqslant C|z_1 - z_2|^\alpha \quad (z_1, z_2 \in \mathbb{D})$$

with some $C = C(f) > 0$. Further, BMOA (respectively VMOA) is the analytic subspace of BMO($\mathbb{T}$) (respectively VMO($\mathbb{T}$)), the space of functions with *bounded* (respectively *vanishing*) *mean oscillation*; see [17, Chapter VI]. Finally, $\mathcal{B}$ stands for the *Bloch space*, defined as the set of analytic functions f with

$$\sup\big\{\big(1 - |z|\big)\big|f'(z)\big|\colon\ z \in \mathbb{D}\big\} < \infty.$$

Some other pairs (X, Y) to be studied are obtained by coupling the above classes with Besov spaces; these will be defined later on.

In fact, among the four inclusions in (1.3), only the first and the last—i.e., only the pairs $(A^\alpha, \mathcal{A})$ and (BMOA, $\mathcal{B}$)—are of interest to us, since the other two lead to trivial classes of inner functions (representing the two extreme situations), as we shall now explain.