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Semiclassical states for nonlinear Schrödinger equations with sign-changing potentials

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Abstract

We establish the existence and multiplicity of semiclassical bound states of the following nonlinear Schrödinger equation:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

where V changes sign and g is superlinear with critical or supercritical growth as $|u| \to \infty$.

Keywords: Nonlinear Schrödinger equation; Sign-changing potential; Superlinear; Supercritical growth

1. Introduction and main results

We consider the existence and multiplicity of semiclassical bound states to the following nonlinear Schrödinger equation:

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

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where $0 < \varepsilon \ll 1$. We are interested in the case where the potential V *changes sign* and the nonlinearity g is superlinear with *critical or supercritical* growth as $|u| \to \infty$. Here we say that V *changes sign* if $V(x_1) < 0 < V(x_2)$ for some $x_1, x_2 \in \mathbb{R}^N$; *g is superlinear* if $|g(x,u)|/|u| \to \infty$ as $|u| \to \infty$; and *g is critical or supercritical* if $N \geqslant 3$ and $c_1|u|^{2^*-1} \leqslant |g(x,u)| \leqslant c_2|u|^{2^*-1}$ or only $c_1|u|^{2^*-1} \leqslant |g(x,u)|$ with $2^* = 2N/(N-2)$ for all large $|u|$. The motivation of such a study is two-fold. On one hand, it is expected that $(\mathcal{P}_\varepsilon)$ has solutions $u \in H^1(\mathbb{R}^N)$ provided, roughly speaking, $\liminf_{|x|\to\infty} V(x) > 0$ (whether or not it changes sign). It is known that by variational arguments the Dirichlet problem on smooth bounded domain $\Omega \subset \mathbb{R}^N$:

$$-\Delta u + V(x)u = |u|^{p-2}u \quad \text{in } \Omega,\ p \in (2, 2^*), \qquad u = 0 \quad \text{on } \partial\Omega$$

always possesses solutions $u \in H_0^1(\Omega)$ without any restriction on the sign of $V(x)$. For the Schrödinger equation $(\mathcal{P}_\varepsilon)$, the condition that $\liminf_{|x|\to\infty} V(x) > 0$ guarantees the embedding

$$\|u\|_{H^1}^2 \leqslant c_\varepsilon \int\limits_{\mathbb{R}^N} \big(\varepsilon^2|\nabla u|^2 + V^+(x)u^2\big), \quad V^+(x) := \max\big\{0, V(x)\big\},$$

hence the variational argument for Dirichlet problem should work. On the other hand, when V changes sign the energy functional associated to the equation is indefinite and consequently has no mountain-pass structure, which stimulates the development of new methods.

Problem $(\mathcal{P}_\varepsilon)$ arises in finding standing wave solutions of the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\varphi + W(x)\varphi - f\big(x, |\varphi|\big)\varphi. \tag{1.1}$$

A standing wave solution of (1.1) is a solution of the form $\varphi(x,t) = u(x)e^{-\frac{iEt}{\hbar}}$. Then $\varphi(x,t)$ solves (1.1) if and only if $u(x)$ solves $(\mathcal{P}_\varepsilon)$ with $V(x) = W(x) - E, \varepsilon^2 = \frac{\hbar^2}{2m}$ and $g(x,u) = f(x,|u|)u$.

Equation $(\mathcal{P}_\varepsilon)$ has being extensively investigated in the literatures based on various assumptions on the potential $V(x)$ and the nonlinearity $g(x,u)$. See for example [1,6,7,11,12,14–22,24, 25] and the references therein. We summarize the findings in the following three cases:

(a) $\inf V > 0$ *and g is superlinear and subcritical*. Most of the papers deal with this case. Floer and Weinstein in [14] considered $N = 1$, $g(u) = u^3$ and studied firstly the existence of single and multiple spike solutions based on a Lyapunov–Schmidt reduction. This result was extended in higher dimension and for $g(u) = |u|^{p-2}u$ in Oh [20,21]. A mountain-pass reduction method has been subsequently applying to finding solutions of $(\mathcal{P}_\varepsilon)$. In [1] Ambrosetti, Badiale and Cingolani studied concentration phenomena of the solutions at isolated local minima and maxima of V with polynomial degeneracy. See also Grossi [15], Li [19] and Pistoia [22] for related results. In Kang and Wei [18] the authors establish the existence of positive solutions with any prescribed number of spikes clustering around a given local maximum point of V. Without assumption of non-degeneracy on critical points of V, the existence of (positive) solutions was handled in del Pino and Felmer [11,12] and Jeanjean and Tanaka [17]. For concentrations on higher-dimensional sets, we refer to Ambrosetti, Malchiodi and Ni [2], and M. del Pino, M. Kowalczyk and Wei [13].