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Dilation of generalized Toeplitz kernels on ordered groups ☆

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Abstract

We introduce the concept of Toeplitz–Kreĭn–Cotlar triplet on ordered groups and we prove a general dilation result and a general representation result for the positive definite case.

This general result includes and extends previous generalizations of the Kreĭn extension theorem, the Sz.-Nagy and Foias commutant lifting theorem and the generalized Herglotz–Bochner–Weil theorem.

Keywords: Ordered group; Positive definite function; Toeplitz kernel; Completely positive map; Dilation; Extension; Commutant lifting

1. Introduction

If Γ is an abelian group and $L(\mathcal{H})$ stands for the space of the bounded linear operators of a Hilbert space $\mathcal{H}$, an $L(\mathcal{H})$-valued kernel is a function $K:\Gamma \times \Gamma \to L(\mathcal{H})$.

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The kernel K is said to be positive definite if

$$\sum_{i,j=1}^{n} \langle K(x_i, y_j)h_i, h_j \rangle_{\mathcal{H}} \geqslant 0$$

whenever n is a natural number, $x_1, \ldots, x_n \in \Gamma$ and $h_1, \ldots, h_n \in \mathcal{H}$.

The kernel K is called Toeplitz if there exists a function $k : \Gamma \to L(\mathcal{H})$ such that $K(x, y) = k(x - y)$ for all $x, y \in \Gamma$.

As it is well known positive definite Toeplitz kernels appear in many problems of analysis.

Also several problems in analysis lead to the consideration of the so-called generalized Toeplitz kernels, K defined on $\mathbb{Z} \times \mathbb{Z}$. These kernels satisfy a more general condition than the Toeplitz one. It is supposed that there exist four functions $(k_{\alpha\beta})_{\alpha,\beta=1,2}$ such that

$$K(x, y) = k_{\alpha\beta}(x - y)$$

whenever $(x, y) \in \mathbb{Z}_\alpha \times \mathbb{Z}_\beta$, where $\mathbb{Z}_1 = \{n \in \mathbb{Z}\colon n \geqslant 0\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z}\colon n < 0\}$. A similar situation occurs for $\mathbb{R} \times \mathbb{R}$.

This notion of generalized Toeplitz kernels was introduced in [16], where a generalization of the Herglotz–Bochner theorem for such kernels and applications to the Helson–Szegö theorem were obtained. In other papers further developments were given, an important part of this development is related with the Sarason interpolation theorem [30], the Sz.-Nagy and Foias commutant lifting theorem [33,34] and the Kreĭn extension theorem [24] (cf. the papers of Arocena, Cotlar, Sadosky, Bruzual and Domínguez in the bibliography).

Several extensions and applications of these two theorems have been given in [4,8,9,11,12,14, 15,18,21,22,25,27]. See also [10] for more references.

A partition similar to $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$ makes sense in general ordered groups $(\Gamma, +)$, thus an analogous notion can be defined in this kind of groups. We are going to introduce a more general concept, the notion of Toeplitz–Kreĭn–Cotlar triplet on ordered groups and we will prove a general dilation result.

This general dilation result include and extends previous generalizations of the Kreĭn extension theorem, the Sz.-Nagy and Foias commutant lifting theorem and the generalized Herglotz–Bochner–Weil theorem. As a tool, the Arveson extension theorem [7] and the Stinespring representation theorem [32] are used.

We also obtain a representation result for weakly measurable Toeplitz–Kreĭn–Cotlar triplets.

The paper is organized as follows. In Section 2 we give some preliminaries definitions and results. In Section 3 we give the definition of Toeplitz–Kreĭn–Cotlar triplets and some related results. In Section 4 we prove our main result. In Section 5 we obtain our main representation result. Finally, in Section 6 we obtain some corollaries, which include and extend some previous results.

2. Preliminaries

Definition 2.1. If Ω is an abelian group, Λ is a subset of Ω and $L(\mathcal{H})$ stands for the space of the bounded linear operators of a Hilbert space $\mathcal{H}$, a function $F : \Lambda \to L(\mathcal{H})$ is said to be *positive*