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A trace theorem for Dirichlet forms on fractals

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Abstract

We consider a trace theorem for self-similar Dirichlet forms on self-similar sets to self-similar subsets. In particular, we characterize the trace of the domains of Dirichlet forms on Sierpinski gaskets and Sierpinski carpets to their boundaries, where the boundaries are represented by triangles and squares that confine the gaskets and the carpets. As an application, we construct diffusion processes on a collection of fractals called fractal fields. These processes behave as an appropriate fractal diffusion within each fractal component of the field.

Keywords: Trace theorem; Self-similar sets; Dirichlet forms; Diffusions on fractals; Lipschitz spaces; Besov spaces; Sierpinski carpets

1. Introduction

The traces of Sobolev spaces on $\mathbb{R}^n$ to linear subspaces have been studied from various viewpoints as the generalizations of the Sobolev imbedding theorem. Further, the extension of Sobolev, Besov, and Lipschitz spaces from subdomains of $\mathbb{R}^n$ to whole spaces has been exten-

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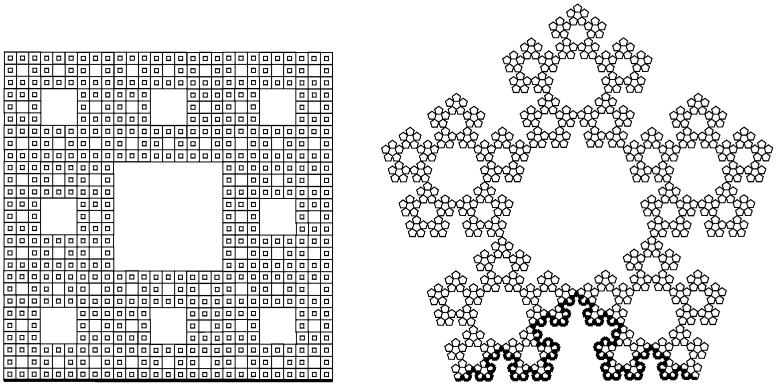

Fig. 1. The Sierpinski carpet and the Pentakun.

sively studied (see, for example, [1,26] and references therein). Since the 1980s, the problems for Besov-type spaces have been generalized for more complicated spaces, namely, the so-called Alfors d-regular sets [16,29].

On the other hand, recent developments in the analysis of fractals shed new light on these problems. Diffusion processes and "Laplace" operators are constructed on fractals such as Sierpinski gaskets and Sierpinski carpets. It is observed that the domains of the corresponding Dirichlet forms are Besov–Lipschitz spaces.

In this paper, we consider the following natural question. Given a Besov-type space on a self-similar fractal K, what is the trace of the space to a self-similar subspace L? Figure 1 shows two examples. The figure on the left is obtained when K is the so-called two-dimensional Sierpinski carpet (see Sections 2 and 5.3 for the definition) and L is the line on the bottom (indicated by the thick line). The figure on the right is obtained when K is the Pentakun (a self-similar fractal determined by five contraction maps; see Section 5.2 for the definition) and L is a Koch-like curve (indicated by the thick curve). In each case, the domain of the Dirichlet form on K is a Besov–Lipschitz space; however, the trace cannot be obtained by using the general theory given by Jonsson, Wallin [16] and Triebel [29].

This problem was recently solved by Jonsson [15] for one typical case, i.e., when K is a two-dimensional Sierpinski gasket and L is the bottom line. However, his methods strongly depend on the structure of the Sierpinski gasket and its Dirichlet form, and they cannot be applied to the so-called infinitely ramified fractals such as the Sierpinski carpets. Instead, we use the self-similarity of the Dirichlet form and the compactness property of a family of harmonic functions, which can be obtained using the elliptic Harnack inequalities. Our methods can be applied to the Sierpinski carpets (even to the higher-dimensional ones), and we can state the trace theorem under some abstract framework. In fact, we would need various assumptions for K and for the Dirichlet form on K, which are stated in Section 2. Unless these conditions are satisfied, nonstandard indices may appear in the trace spaces because of the "complexity" of the space (see Section 5.4 for an example).

In order to prove our trace theorem, we provide a discrete approximation of the Besov–Lipschitz space in Section 3.1. The approximation result is new, and it is regarded as a generalization of the main result in [17]. The restriction theorem is given in Section 3.2; the key estimate (Proposition 3.8) is based on the idea used by one of the authors in [13]. The extension