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Spectral multipliers on a class of *NA* groups with rank two

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Abstract

We consider a family of real *NA* groups with rank two, and we prove that these groups have sub-Laplacians with differentiable L^p functional calculus for all $p \geqslant 1$.

Keywords: Solvable Lie group; Exponential growth of the volume; Sub-Laplacian; Spectral multiplier; Differentiable functional calculus

0. Introduction

Let G be a Lie group, and X_j, $j = 1, \dots, d$, be left-invariant vector fields on G which generate the Lie algebra of G. Consider the left-invariant sub-Laplacian $\Delta = -\sum_{j=1}^{d} X_j^2$. It is a formally self-adjoint and non-negative operator on the space $L^2(G)$ relative to the right-invariant Haar measure on G. From the spectral theorem, every bounded Borel function m on $[0, \infty)$ determines a bounded operator $m(\Delta)$ on $L^2(G)$ via the formula $m(\Delta) = \int_0^\infty m(\lambda)\, dE_\lambda$, where $\Delta = \int_0^\infty \lambda\, dE_\lambda$ is the spectral resolution of Δ. A problem that arises naturally is that of finding sufficient conditions on the function m, so that the operator $m(\Delta)$ extends to a bounded operator on $L^p(G)$ for some $p \neq 2$. This was given a lot of attention over the past decades, for various

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types of Lie groups G. In this paper, we interest ourselves to the case where G is solvable and has exponential volume growth.

The literature shows that there are (at least) two classes of solvable Lie groups with exponential volume growth and invariant sub-Laplacians: groups with sub-Laplacians which admit differentiable L^p functional calculus (see, e.g., Hebisch [7–9], Cowling et al. [2], Mustapha [14], Hebisch and Steger [10], David-Guillou [4], Gnewuch [6]) , and groups with sub-Laplacians of holomorphic L^p type (see Christ and Müller [1], Ludwig and Müller [13], Hebisch et al. [11]). Here we prove a multiplier theorem for groups and sub-Laplacians of the first class.

Before we describe our result, let us point out that all the solvable Lie groups that have been shown to have sub-Laplacians with differentiable L^p functional calculus for some $p \neq 2$ are *NA* groups, most of them with rank one. Indeed, only two types of *NA* groups with rank greater than one are known to have sub-Laplacians with this property: Iwasawa *NA* groups coming from arbitrary semisimple Lie groups (see [2,7]), and metabelian groups (see [9]). The point of this paper is to investigate further the high rank case; we give a family of *NA* groups with rank two, on which distinguished sub-Laplacians have differentiable L^p functional calculus for every $p \geqslant 1$.

1. Main results

1.1. Statement of the multiplier theorem and related L^1 estimate on the heat kernel

Let H be a stratified Lie group, that is a real connected simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{h}$ has a vector space decomposition

$$\mathfrak{h} = \bigoplus_{j=1}^{J} V_j,$$

such that

$$[V_1, V_j] = V_{j+1}, \quad j = 1, \dots, J-1. \tag{1}$$

We consider the family of algebra automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ of $\mathfrak{h}$ defined by

$$\sigma_t(X_j) = \mathrm{e}^{jt} X_j, \quad X_j \in V_j.$$

Let $\exp_H$ be the exponential map of N. The maps $\exp_H \circ \sigma_t \circ \exp_H^{-1}$ are group automorphisms of H, called *dilations* on H; we denote them also by σ_t. Endowed with those dilations, H is said to be *homogeneous group* with *homogeneous dimension*

$$Q = \sum_{j=1}^{J} j \dim(V_j).$$

Fix $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}^n$. Let G be the semidirect product

$$G = G_\alpha = (H \times \mathbb{R}) \ltimes \mathbb{R}^n,$$