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Commutative C^*-algebras of Toeplitz operators and quantization on the unit disk ☆

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Abstract

A family of recently discovered commutative C^*-algebras of Toeplitz operators on the unit disk can be classified as follows. Each pencil of hyperbolic straight lines determines a set of symbols consisting of functions which are constant on the corresponding cycles, the orthogonal trajectories to lines forming a pencil. The C^*-algebra generated by Toeplitz operators with such symbols turns out to be commutative. We show that these cases are the only possible ones which generate the commutative C^*-algebras of Toeplitz operators on each weighted Bergman space.

Keywords: Weighted Bergman spaces; Toeplitz operator; Commutative C^*-algebra; Pencil of hyperbolic geodesics; Berezin quantization

1. Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Consider $L_2(\mathbb{D})$ with respect to standard normalized measure and its subspace, the Bergman space $\mathcal{A}^2(\mathbb{D})$, which consists of functions analytic in $\mathbb{D}$. Let $B_{\mathbb{D}}$ stand for the orthogonal Bergman projection of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$. Given a function $a(z) \in L_\infty(\mathbb{D})$, the Toeplitz operator T_a with *symbol* $a = a(z)$ is defined on $\mathcal{A}^2(\mathbb{D})$ as follows:

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$$T_a : \varphi \in \mathcal{A}^2(\mathbb{D}) \mapsto B_{\mathbb{D}}(a\varphi) \in \mathcal{A}^2(\mathbb{D}).$$

It is well known (see, for example, [15]) that Toeplitz operators with radial symbols commute and can be diagonalized in the standard monomial basis in $\mathcal{A}^2(\mathbb{D})$. Surprisingly (see, for details, [18,19]) aside from radial symbols there exists a rich family of symbols which generate *commutative* C^*-algebras of Toeplitz operators. Moreover, it turns out that these commutative properties do not depend at all on smoothness properties of symbols: the corresponding symbols can be merely measurable. The prime cause here appears to be the geometric configuration of level lines of symbols. All commutative C^*-algebras of Toeplitz operators discovered can be classified by pencils of geodesics on the unit disk, considered as the hyperbolic plane. More precise, given a pencil of geodesics, consider the set of symbols constant on the corresponding cycles, the orthogonal trajectories to geodesics forming the pencil. The C^*-algebra generated by Toeplitz operators with such symbols turns to be commutative. It was shown later (see [8–10]) that the same classes of symbols generate commutative C^*-algebras of Toeplitz operators on *each weighted* Bergman space.

At the same time the principal question,

> *whether the above classes are the only possible sets of symbols which might generate the commutative C^*-algebras of Toeplitz operators on each weighted Bergman space*,

has remained open.

There is a trivial case having in fact no connection with specific properties of Toeplitz operators. Each C^*-algebra with identity (Toeplitz operators with the symbol $e(z) \equiv 1$) generated by a self-adjoint element (Toeplitz operator with a real-valued symbol $a = a(z)$) is obviously commutative. We exclude this obvious case from the further considerations.

The aim of the paper is to give the affirmative answer to the above question.

The commutativity of the above algebras on *each* weighted Bergman space is of great importance and permits us to make use of the Berezin quantization procedure (see, for example, [2,3]). At the same time to obtain the necessary information about potential symbols we need to calculate the *second and third terms* in the asymptotic expansion of a commutator. It turns out that the first three terms of this expansion together provide us with exact geometric information: *in order to generate a commutative C^*-algebra of Toeplitz operators on each weighted Bergman space the symbols must be constant on the cycles of a pencil of geodesics*.

At the same time we show that there exist, in a sense non-typical, C^*-algebras of Toeplitz operators commutative *only on a single* Bergman space.

The paper is organized as follows.

In Section 2 we introduce Toeplitz operators on the weighted Bergman spaces as well as the pencils of geodesics, and sketch the proof, for a parabolic case, that the C^*-algebra generated by Toeplitz operators, whose symbols are constant on cycles, is commutative in each weighted Bergman space.

In Section 3 we discuss the symbol classes, clarifying that they have to be linear sets of smooth functions closed under complex conjugation and containing $e(z) \equiv 1$.

In Section 4 we show that there exist C^*-algebras of Toeplitz operators commutative on a *single* weighted Bergman space. However, in the examples given the set of generating symbols is quite restricted.

In Section 5 we recall necessary facts on Berezin quantization on the hyperbolic plane and give the three-term asymptotic expansion of a commutator (of Wick symbols) of Toeplitz operators.