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The Lyapunov function for Schrödinger operators with a periodic 2×2 matrix potential

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Abstract

We consider the Schrödinger operator on the real line with a 2×2 matrix-valued 1-periodic potential. The spectrum of this operator is absolutely continuous and consists of intervals separated by gaps. We define a Lyapunov function which is analytic on a two-sheeted Riemann surface. On each sheet, the Lyapunov function has the same properties as in the scalar case, but it has branch points, which we call resonances. We prove the existence of real as well as non-real resonances for specific potentials. We determine the asymptotics of the periodic and the anti-periodic spectrum and of the resonances at high energy. We show that there exist two type of gaps: (1) stable gaps, where the endpoints are the periodic and the anti-periodic eigenvalues, (2) unstable (resonance) gaps, where the endpoints are resonances (i.e., real branch points of the Lyapunov function). We also show that periodic and anti-periodic spectrum together determine the spectrum of the matrix Hill operator.

Keywords: Schrödinger operator; Periodic matrix potentials; Spectral bands; Spectral gaps

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1. Introduction and main results

We consider the self-adjoint operator $Ty = -y'' + V(x)y$, acting in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where V is a symmetric 1-periodic 2×2 matrix potential which belongs to the real space $\mathcal{H}^p$, $p = 1, 2$, given by

$$\mathcal{H}^p = \left\{ V = V^* = V^T = \begin{pmatrix} V_1 & V_3 \\ V_3 & V_2 \end{pmatrix} : \int_0^1 V_3(x)\, dx = 0 \right\},$$

equipped with the norm $\|V\|_p^p = \int_0^1 (|V_1(x)|^p + |V_2(x)|^p + 2|V_3(x)|^p)\, dx < \infty$. Without loss of generality *we assume*

$$V_{(0)} = \int_0^1 V(t)\, dt, \qquad V_{10} \leqslant V_{20}, \qquad V_{30} = 0, \qquad V_{m0} = \int_0^1 V_m(x)\, dx, \quad m = 1, 2, 3.$$

Let us introduce the self-adjoint operator $T^0 = -d^2/dx^2$, with the domain $\mathrm{Dom}(T^0) = W_2^2(\mathbb{R}) \oplus W_2^2(\mathbb{R})$. In order to get self-adjointness of T we use symmetric quadratic forms. We briefly recall a well-known argument (see [19]). We define the form $(V\psi, \psi_1) = -\int_{\mathbb{R}} V\psi\overline{\psi}_1\, dx$, $\psi, \psi_1 \in \mathrm{Dom}(T^0)$. Using the estimate (see [13])

$$\big|(q'f, f)\big| < \varepsilon(f', f') + b_\varepsilon(f, f) \quad \text{for any small } \varepsilon > 0 \text{ and some } b_\varepsilon > 0$$
$$\text{and for any } f \in W_2^2(\mathbb{R}),\ q \in L^2(\mathbb{R}/\mathbb{Z}),$$

we deduce that

$$\big|(V\psi, \psi)\big| < (1/2)(\psi', \psi') + b(\psi, \psi), \quad \psi \in W_2^2(\mathbb{R}) \oplus W_2^2(\mathbb{R}).$$

Thus we can apply the KLMN theorem (see [19]) to define $T = -d^2/dx^2 + V$. There exists a unique self-adjoint operator T with form domain $\mathcal{Q}(T) = W_1^2(\mathbb{R}) \oplus W_1^2(\mathbb{R})$ and

$$(T\psi, \psi_1) = (-\psi'', \psi_1) + (V\psi, \psi_1) \quad \text{all } \psi, \psi_1 \in \mathcal{Q}\big(T^0\big) = W_1^2(\mathbb{R}) \oplus W_1^2(\mathbb{R}).$$

Any domain of essential self-adjointness for T^0 is a form core for T.

It is well known (see [6, pp. 1486–1494], [8]) that the spectrum $\sigma(T)$ of T is absolutely continuous and consists of non-degenerate intervals S_n, $n = 1, 2, \ldots$. These intervals are separated by gaps G_n with lengths $|G_n| > 0$, $n = 1, 2, \ldots, N_G \leqslant \infty$. Introduce the fundamental 2×2 matrix solutions $\varphi(x, \lambda)$, $\vartheta(x, \lambda)$ of the equation

$$-y'' + V(x)y = \lambda y, \quad \lambda \in \mathbb{C}, \tag{1.1}$$