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The Godbillon–Vey cyclic cocycle for PL-foliations ☆

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Abstract

We define a cyclic cocycle which corresponds to the piecewise linear Godbillon–Vey class of Ghys and Sergiescu [E. Ghys, V. Sergiescu, Sur un groupe remarquable de difféomorphismes du cercle, Comment. Math. Helv. 62 (1987) 185–239]. Using Connes's pairing [A. Connes, Non-commutative differential geometry. Part II: De Rham homology and noncommutative algebra, Publ. Math. Inst. Hautes Études Sci. 62 (1985) 257–360; A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, in: H. Araki, G. Effros (Eds.), Geometric Methods in Operator Algebras, Pitman Res. Notes Math. Ser., vol. 123, Longman, Harlow, 1986, pp. 52–144] between cyclic cohomology and K-theory, we then evaluate this cocycle on a suitable K-theory class and obtain a nontrivial result, for foliations of the 3-torus by slope components.

Keywords: Foliations; C^*-Algebras; K-Theory; Godbillon–Vey invariant; Piecewise linear homeomorphisms

1. Introduction

The aim of this paper is to extend the definition of the Godbillon–Vey cyclic cocycle, given by Connes in [6], to foliations whose transverse structure is not smooth, and in particular to transversely piecewise linear foliations, which we will call in short PL-foliations.

The Godbillon–Vey invariant was first defined geometrically in [9] for smooth (or at least class C^2) foliations. Afterwards, a discrete version for PL-foliations was given by Ghys and Sergiescu in [8]. In this piecewise linear context, there are many very simple concrete exam-

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ples of foliations for which the invariant is nonzero, whereas in the smooth case the nontrivial examples are more complicated (Roussarie and Thurston [18]).

A more general version of the Godbillon–Vey invariant was given by Tsuboi in [19], in particular for foliations which are transversely "of class P." Our aim is to see how the theory of noncommutative geometry applies to this extended class of foliations, and this will enable us to compute some simple nontrivial examples.

In [6], Connes defined a cyclic 2-cocycle corresponding to the Godbillon–Vey class for smooth foliations. The framework for this definition can be summarized as follows.

Definition 1. A cyclic 2-cocycle on a $\mathbf{C}$-algebra B is a trilinear form τ satisfying the two following conditions: $\forall x^0, x^1, x^2, x^3 \in B$,

$$\tau(x^0x^1, x^2, x^3) - \tau(x^0, x^1x^2, x^3) + \tau(x^0, x^1, x^2x^3) - \tau(x^3x^0, x^1, x^2) = 0,$$
$$\tau(x^0, x^1, x^2) = \tau(x^2, x^0, x^1).$$

Every cyclic 2-cocycle on B induces an algebraic K-theory map $K_0(B) \to \mathbf{C}$, which we denote by $\langle \tau, e\rangle = \tau\#\mathrm{Tr}(e, e, e)$ because it is the cup product of τ with the usual trace on the complex matrix space, applied to an idempotent e [5].

However, for the Godbillon–Vey class on a manifold with a smooth foliation $(V, \mathcal{F})$, the cyclic cocycle is not defined on the whole foliation C^*-algebra, but on the dense subalgebra $B \subset C^*(V, \mathcal{F})$ of smooth functions with compact support. Therefore, the cocycle will be interesting only if it satisfies a sufficient "continuity condition" so that its associated K-theory map extends to a *global* K-theory map $K_0(C^*(V, \mathcal{F})) \to \mathbf{C}$. This occurs especially in the case of a 2-trace:

Definition 2. Let A be a Banach algebra and $B \subset A$ a dense subalgebra. A 2-trace τ on A with domain B is a cyclic 2-cocycle on B such that:

$$\forall a^1, a^2 \in B, \exists C \geqslant 0, \forall x^1, x^2 \in B, \quad \left|\tau(x^1, a^1x^2, a^2) - \tau(x^1a^1, x^2, a^2)\right| \leqslant C\|x^1\|\|x^2\|.$$

Theorem 2.7 of [6] says that the algebraic K-theory map associated to the 2-trace τ extends to a global K-theory map $K_0(A) \to \mathbf{C}$.

The main point in this theorem is the fact that a 2-trace extends to a subalgebra $\mathcal{B}$ ($B \subset \mathcal{B} \subset A$), which is stable under holomorphic functional calculus in A. Karoubi's extension theorem [14] then ensures that the inclusion $\mathcal{B} \to A$ induces a K-theory isomorphism.

Wa can now state the main result of this paper.

Theorem 3. (Theorem 25) *Let Γ be a discrete subgroup of homeomorphisms of class P of S^1, $G = (S^1 \rtimes \Gamma)/\sim$ the holonomy groupoid, and ω the generalized Godbillon–Vey group 2-cocycle as in* [19]. *The following is a densely defined cyclic 2-cocycle, which induces a 2-trace τ on $C_r^*(G)$:*

$$\tilde{\tau}(f^0, f^1, f^2) = \int_{\gamma_0\gamma_1\gamma_2 \in G^{(0)}} f^0(\gamma_0) f^1(\gamma_1) f^2(\gamma_2)\omega(\gamma_1, \gamma_2).$$