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On the BKS pairing for Kähler quantizations of the cotangent bundle of a Lie group

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Abstract

A natural one-parameter family of Kähler quantizations of the cotangent bundle T^*K of a compact Lie group K, taking into account the half-form correction, was studied in [C. Florentino, P. Matias, J. Mourão, J.P. Nunes, Geometric quantization, complex structures and the coherent state transform, J. Funct. Anal. 221 (2005) 303–322]. In the present paper, it is shown that the associated Blattner–Kostant–Sternberg (BKS) pairing map is unitary and coincides with the parallel transport of the quantum connection introduced in our previous work, from the point of view of [S. Axelrod, S. Della Pietra, E. Witten, Geometric quantization of Chern–Simons gauge theory, J. Differential Geom. 33 (1991) 787–902]. The BKS pairing map is a composition of (unitary) coherent state transforms of K, introduced in [B.C. Hall, The Segal–Bargmann coherent state transform for compact Lie groups, J. Funct. Anal. 122 (1994) 103–151]. Continuity of the Hermitian structure on the quantum bundle, in the limit when one of the Kähler polarizations degenerates to the vertical real polarization, leads to the unitarity of the corresponding BKS pairing map. This is in agreement with the unitarity up to scaling (with respect to a rescaled inner product) of this pairing map, established by Hall.

Keywords: Blattner–Kostant–Sternberg pairing; Geometric quantization; Coherent state transform for Lie groups

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Contents

1. Introduction

Let K be a compact, connected Lie group of dimension n and let T^*K be its cotangent bundle. We start by recalling some aspects of [2] where, in connection with work of Hall in [6], the geometric quantization of T^*K was studied using a natural one-parameter family of Kähler structures. These Kähler structures are induced on T^*K via the following natural identifications of T^*K with the complexified group $K_{\mathbb{C}}$. Consider, for any real parameter $s > 0$, the diffeomorphisms

$$\begin{aligned} \psi_s : T^*K &\to K_{\mathbb{C}} \\ (x, Y) &\mapsto \psi_s(x, Y) = xe^{isY}. \end{aligned} \tag{1.1}$$

Here, $x \in K$, $Y \in \mathfrak{K} \equiv \mathrm{Lie}(K)$, and we identify T^*K with $K \times \mathfrak{K}^*$ using left invariant forms and then with $K \times \mathfrak{K}$ by means of a fixed Ad-invariant inner product on $\mathfrak{K}$. The diffeomorphisms ψ_s endow T^*K with a family of complex structures J_s and one can check that, together with the canonical symplectic structure ω on T^*K, the pair (ω, J_s) defines a Kähler structure on T^*K for every $s \in \mathbb{R}_+$ [2]. This family includes the Kähler structure on T^*K considered by Hall in [6].

In this paper, we consider the Blattner–Kostant–Sternberg (BKS) pairing between two different Kähler quantizations of T^*K. To describe the results, let us consider the framework used in [2]. Let L denote the trivial complex line bundle on T^*K, with trivial Hermitian structure (its sections are therefore identified with C^∞ functions on T^*K). Following the geometric quantization program with half-form correction, let us introduce the half-form bundle δ_s, which is a square root of the (trivial) J_s-canonical bundle κ_s over T^*K. Choosing canonical trivializing J_s-holomorphic sections Ω_s of $\kappa_s = \delta_s^2$ and $\sqrt{\Omega_s}$ of δ_s (we refer to the next section for precise formulas) one introduces a natural Hermitian structure on $L \otimes \delta_s$ so that, for a smooth section σ_s of the form

$$\sigma_s = f\sqrt{\Omega_s}, \tag{1.2}$$

with $f \in C^\infty(T^*K)$, one has

$$|\sigma_s|^2 := |f|^2 |\Omega_s|, \tag{1.3}$$

where $|\Omega_s|$ is defined by $\overline{\Omega}_s \wedge \Omega_s = |\Omega_s|^2 b\epsilon$, $b = (2i)^n(-1)^{n(n-1)/2}$ and $\epsilon = \frac{1}{n!}\omega^n$ is the Liouville measure on T^*K.