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Eigenvalue gaps for the Cauchy process and a Poincaré inequality

Rodrigo Bañuelos [a,*,1], Tadeusz Kulczycki [b,2]

[a] *Department of Mathematics, Purdue University, West Lafayette, IN 47906, USA*
[b] *Institute of Mathematics, Wrocław University of Technology, 50-370 Wrocław, Poland*

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Abstract

A connection between the semigroup of the Cauchy process killed upon exiting a domain D and a mixed boundary value problem for the Laplacian in one dimension higher known as *the mixed Steklov problem*, was established in [R. Bañuelos, T. Kulczycki, The Cauchy process and the Steklov problem, J. Funct. Anal. 211 (2004) 355–423]. From this, a variational characterization for the eigenvalues λ_n, $n \geqslant 1$, of the Cauchy process in D was obtained. In this paper we obtain a variational characterization of the difference between λ_n and λ_1. We study bounded convex domains which are symmetric with respect to one of the coordinate axis and obtain lower bound estimates for $\lambda_* - \lambda_1$ where λ_* is the eigenvalue corresponding to the "first" antisymmetric eigenfunction for D. The proof is based on a variational characterization of $\lambda_* - \lambda_1$ and on a weighted Poincaré-type inequality. The Poincaré inequality is valid for all α symmetric stable processes, $0 < \alpha \leqslant 2$, and any other process obtained from Brownian motion by subordination. We also prove upper bound estimates for the spectral gap $\lambda_2 - \lambda_1$ in bounded convex domains.

Keywords: Cauchy process; Steklov problem; Spectral gap; Poincaré inequality

* Corresponding author.
E-mail addresses: banuelos@math.purdue.edu (R. Bañuelos), tkulczyc@im.pwr.wroc.pl (T. Kulczycki).

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Contents

1. Introduction

The spectral gap estimates for eigenvalues of the Laplacian with Dirichlet boundary conditions, henceforth referred to as the *Dirichlet Laplacian*, have attracted considerable attention for many years [2,3,10,29,36,37,40]. The Dirichlet Laplacian is the infinitesimal generator of the semigroup of Brownian motion killed upon leaving a domain. Therefore questions concerning eigenvalues of this operator can be studied both by analytic and probabilistic methods. The question of precise lower bounds for the spectral gap for the Dirichlet Laplacian (the difference between the first two eigenvalues) was raised by M. van den Berg [10] (see also Yau [38, problem No. 44]) and was motivated by problems in mathematical physics related to the behavior of free Boson gases. The conjecture, which remains open, asserts that for any convex bounded domain D of diameter d_D, the spectral gap is bounded below by $3\pi^2/d_D^2$. (See [5,9,21] where some special cases of the conjecture are proved and [22,39] for more general "partition function" inequalities.) The spectral gap has also been studied for the Laplacian with Neumann boundary conditions and for Schrödinger operators [2,33,36,37]. From the probabilistic point of view, the spectral gap for the Dirichlet Laplacian determines the rate to equilibrium for the Brownian motion conditioned to remain forever in D, the *Doob h-process* corresponding to the ground state eigenfunction.

The natural question arises as to whether these results can be extended to other non-local, pseudo-differential operators. The class of such operators which are most closely related to the Laplacian Δ from the point of view of Brownian motion are $-(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. These are the infinitesimal generators of the symmetric α-stable processes. These processes do not have continuous paths which is related to non-locality of $-(-\Delta)^{\alpha/2}$. As in the case of Brownian motion, we can consider the semigroup of these processes killed upon exiting domains and we can consider the eigenvalues of such semigroup. Here again, the spectral gap determines the asymptotic exponential rate of convergence to equilibrium for the process conditioned to remain forever in the domain. Instead of speaking of the eigenvalue gap for the operator $-(-\Delta)^{\alpha/2}$ we will very often refer to it as the eigenvalue gap for the corresponding process.

The purpose of this paper is to obtain eigenvalue gap estimates for the Cauchy process, the symmetric α-stable process for $\alpha = 1$. This is done using the connection (established in [6]) between the eigenvalue problem for the Cauchy process and the *mixed Steklov problem*. Both, the methods and the results, are new. The results raise natural questions concerning spectral gaps for other symmetric α-stable processes and for more general Markov processes. We believe that as with the results in [6] which have motivated subsequent work by others, see [18,23,24], the current results will also be of interest. Let X_t be a symmetric α-stable process in $\mathbb{R}^d$, $\alpha \in (0,2]$. This is a process with independent and stationary increments and characteristic function $E^0 e^{i\xi X_t} = e^{-t|\xi|^\alpha}$, $\xi \in \mathbb{R}^d$, $t > 0$. E_x, P_x denote the expectation and probability of this process starting at x, respectively. By $p^{(\alpha)}(t,x,y) = p_t^{(\alpha)}(x-y)$ we will denote the transition