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Interactions

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Abstract

Given a C*-algebra B, a closed *-subalgebra $A \subseteq B$, and a partial isometry S in B which *interacts* with A in the sense that $\mathcal{S}^* a \mathcal{S} = \mathcal{H}(a)\mathcal{S}^*\mathcal{S}$ and $\mathcal{S} a \mathcal{S}^* = \mathcal{V}(a)\mathcal{S}\mathcal{S}^*$, where $\mathcal{V}$ and $\mathcal{H}$ are positive linear operators on A, we derive a few properties which $\mathcal{V}$ and $\mathcal{H}$ are forced to satisfy. Removing B and $\mathcal{S}$ from the picture we define an *interaction* as being a pair of maps $(\mathcal{V}, \mathcal{H})$ satisfying the derived properties. Starting with an abstract interaction $(\mathcal{V}, \mathcal{H})$ over a C*-algebra A we construct a C*-algebra B containing A and a partial isometry S whose *interaction* with A follows the above rules. We then discuss the possibility of constructing a *covariance algebra* from an interaction. This turns out to require a generalization of the notion of correspondences (also known as Pimsner bimodules) which we call a *generalized correspondence*. Such an object should be seen as an usual correspondence, except that the inner-products need not lie in the coefficient algebra. The covariance algebra is then defined using a natural generalization of Pimsner's construction of the celebrated Cuntz–Pimsner algebras.

Keywords: Partial isometry; Generalized correspondence; Ternary rings of operators; Crossed-product; Redundancy; Transfer operator

1. Introduction

In [6] we have introduced a crossed-product construction (see also [7,8]) given an endomorphism α of a C*-algebra A and a transfer operator $\mathcal{L}$, attempting to improve and extend previous constructions. Since then a wealth of interesting examples have been discovered, notably Watatani's study of polynomial maps on subsets of the Riemann sphere [10] and Deaconu's

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generalization of our construction to semigroups of endomorphisms and his study of higher rank graphs [4] (see also [9]).

Recall that, among other things, the crossed-product referred to above involves a partial isometry $\mathcal{S}$ such that

$$\mathcal{S}a = \alpha(a)\mathcal{S} \quad \text{and} \quad \mathcal{S}^* a\mathcal{S} = \mathcal{L}(a), \quad \forall a \in A.$$

If one looks at these relations from a purely aesthetical point of view he or she will likely be bothered by their asymmetry. For instance, $\mathcal{S}$ is allowed to jump from left to right but not vice versa, that is, there is no relation telling us what to do with an expression of the form $a\mathcal{S}$.

Perhaps the cause for this asymmetry is the fact that time evolution is often irreversible. In order to explain this recall that while α can be interpreted as accounting for the future[2] evolution of the system under analysis, the transfer operator is needed to account for the past. In the case of *irreversible systems*, namely systems in which α is not invertible, it is natural that time evolution should behave quite differently depending on whether we are moving forward or backward in time.

If we are speaking of a classical irreversible system, say a continuous (non-invertible) map

$$T : X \to X,$$

where X is a compact space, the trouble with accounting for the past corresponds to the fact that a point x in X may have more than one pre-image under T. Yet the set $T^{-1}(x)$ of all pre-images is always well defined and in some cases one can attach a probability distribution to $T^{-1}(x)$ representing a guess as to what the past looked like. By looking at an already extinguished camp fire it is impossible to tell how did it look like the night before but it is sometimes possible to guess!

Suppose that one indeed is given a probability distribution μ_x on $T^{-1}(x)$ for each x in X. Then, given an observable, i.e. a continuous scalar valued function f on X, one may define

$$\mathcal{L}(f)|_x = \int_{T^{-1}(x)} f(y)\, d\mu_x(y), \quad \forall x \in X.$$

In this way $\mathcal{L}(f)$ represents the expected value of the observable f one unit of time into the past. Supposing that $\mathcal{L}(f)$ is continuous for every f one checks without difficulty that $\mathcal{L}$ is a transfer operator for the endomorphism α of $C(X)$ defined by

$$\alpha : f \in C(X) \mapsto f \circ \alpha \in C(X).$$

In this paper we wish to take a first step toward the study of systems whose future behavior presents the same degree of uncertainty as its past.

Inspired by our previous crossed-product construction we postulate that our given algebra of observables A should be embedded in a larger algebra B (roughly playing the role of the crossed-product) containing a partial isometry $\mathcal{S}$ which governs time evolution. Time evolution itself will

[2] In Dynamical Systems one often thinks of the given map as representing the time evolution and hence its positive iterates represent the "future."